II MSc MATHEMATICS

SUBJECT CODE : 7MMA3C3

SUBJECT NAME : PROBABILITY AND STATISTICS

HANDLED BY

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II YEAR–III SEMESTER
COURSE CODE: 7MMA3C3

CORE COURSE-XI – PROBABILITY AND STATISTICS

Unit I

Probability and Distribution: Introduction – Set theory – The probability set function – Conditional probability and independence – Random variables of the discrete type – Random variables of the continuous type – properties of the distribution function – expectation of random variable – some special expectations – Chebyshev's Inequality.

Unit II

Multivariate Distributions: Distributions of two random variables – Conditional Distributions and Expectations – the correlation coefficient – Independent random variables – extension to several Random variables.

Unit III

Some special Distributions: The Binomial and Related Distributions – The Poisson Distribution– The Gamma and Chi-square Distributions – The Normal Distribution – The Bivariate Normal Distribution.

Unit IV

Distributions of functions of Random variables: Sampling Theory – Transformations of variables of the discrete type – Transformations of variables of the continuous type – the Beta, t and F distributions – Extensions of the change – of – variable Technique – Distributions of order statistics – The Moment generating – Function, Techniques – The distributions of $\overline{X}$ and ns^2/σ^2 – Expectations of functions of Random variables

Unit V

Limiting Distributions : Convergence in distribution – convergence in probability – Limiting Moment Generating Functions – The Central Limit Theorem – Some theorems on Limiting Distributions.

Text Book:

1. Introduction to Mathematical Statistics, (Fifth edition) by Robert V.Hogg and AllenT. Craig Pearson Education Asia.

Chapters I, II, III, IV (Omit 4.10) & V.

Books for Supplementary Reading and Reference:

1. M.Fisz, Probability, Theory and Mathematical Statistics, John Wiley and Sons, New York. 1963.
2. V.K.Rohatgi, An Introduction to Probability Theory and Mathematical Statistics, Wiley Eastern Ltd., New Delhi, 1988 (3rd Print)

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UNIT - I

PROBABILITY AND DISTRIBUTION

* Introduction
* Set theory
* The probability set function
* Conditional probability and independence
* Random variables of the discrete type
* Random variables of the continuous type
* properties of the distribution function
* Expectation of random variable
* Some special expectations
* Chebyshev's inequality.

likelyhood of events happen

Unit-I

probability and distributions

Random experiment:

An experiment can be ~~replac~~ repeated under the same condition, this is called a random experiment.

Sample space:

The collection of every possible outcome is called the experimental space or sample space.

Event:

no of favourable events / Total no of events

Any subset of a sample space S is called an event.

Set theory:

$\frac{1}{1/2}$ no of events : 2^n

A set is a collection of all well-defined objects.

Subset:

If each element of a set c_1 is also an element of a set c_2, the set c_1 is called a subset of the set c_2. It is written as $c_1 \subset c_2$

proper subset?

If c_1 is a proper subset of c_2 and also c_2 is a proper subset of c_1, the two sets have the same elements and this is written as $c_1 = c_2$.

Ex:

Let $c_1 = \{x : 0 \leq x \leq 1\}$ and $c_2 = \{x : -1 \leq x \leq 2\}$
Then c_1 is a proper subset of c_2.

Null set:

If a set c has no elements, c is called a null set. It is written as $c = \phi$

Union:

The set of all elements that belong to atleast one of the sets c_1 and c_2 is called the union of c_1 and c_2. It is written as $c_1 \cup c_2$.

Ex:

Let $c_1 = \{x_1 : 0 \leq x \leq 1\}$ and $c_2 = \{x_2 : -1 \leq x \leq 1\}$

Then $c_1 \cup c_2 = c_2$

Intersection:

The set of all elements that belong to each of the sets c_1 and c_2 is called the intersection of c_1 and c_2. It is written as $c_1 \cap c_2$

Ex:

Let $c_1 = \{(0,0), (1,1), (0,1)\}$ and $c_2 = \{(1,1), (2,2), (1,2)\}$

Then $c_1 \cap c_2 = \{(1,1)\}$

complement:

Let ζ denote a space and let c be the subset of ζ. The set that consists of all elements of ζ that are not elements of c is called the complement of c. It is denoted by c^c.

Ex:

Let $\zeta = \{0, 1, 2, 3, 4\}$ and $c = \{0, 1\}$. Then the complement of c is $c^c = \{2, 3, 4\}$

The probability set function

Defn: (σ - field)

2m Let ξ (zeta) denote the sample space. Let $\mathcal{B}$ be a collection of subsets of ξ. We say $\mathcal{B}$ is a σ- field if

i) $\phi \in B$ (B is not empty)

ii) If $C \in B$ then $C^c \in B$ (B is closed under complements)

iii) If the sequence of the sets $\{C_1, C_2, \dots\}$ is in B than $\bigcup_{i=1}^{\infty} C_i \in B$ (B is closed under countable unions

Defn: probability set function

Let ξ be a sample space and B be a σ- field on ξ. Let P be a real valued function defined on B. Then P is a probability set function if P satisfies the following three conditions:

i) $P(C) \geq 0 \quad \forall\ C \in B$

ii) $P(\xi) = 1$ (sample space)

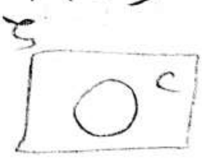
iii) If $\{C_n\}$ is a sequence of sets in B and $C_i \cap C_j = \phi$ for all $i \neq j$ then

$$P(C_1 \cup C_2 \cup \dots) = P(C_1) + P(C_2) + \dots$$

Thm: 1 2m

For each $C \in B$, $P(C) = 1 - P(C^c)$

Proof:-



We have $C \cup C^c = \xi$ and $C \cap C^c = \phi$

By defn of the probability set function

$$P(C \cup C^c) = P(\xi)$$
$$P(C) + P(C^c) = 1$$
$$P(C^c) = 1 - P(C)$$
$$P(C) = 1 - P(C^c)$$

Thm: 2

The probability of the null set is zero ie., $P(\phi) = 0$

2m

soln:-

Let c be a null set ie., $c = \phi$

W.K.T $P(C) = 1 - P(C^c)$ [By thm: 1]

$$P(\phi) = 1 - P(\xi)$$
$$= 1 - 1$$
$$= 0$$

Thm: 3

If C_1 and C_2 are events such that $C_1 \subset C_2$ then $P(C_1) \leq P(C_2)$

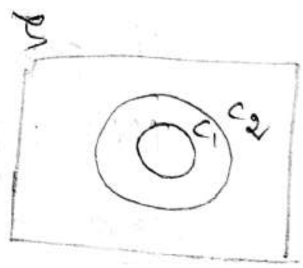
proof:-

Let C_1 and C_2 be the subsets of $\xi \ni : C_1 \subset C_2$

2m

We have $C_2 = C_1 \cup (C_1^c \cap C_2)$ and $C_1 \cap (C_1^c \cap C_2) = \phi$

$$P(C_2) = P(C_1) + P(C_1^c \cap C_2)$$
$$P(C_2) - P(C_1) = P(C_1^c \cap C_2)$$
$$\geq 0$$
$$\therefore P(C_2) \geq P(C_1)$$
$$P(C_1) \leq P(C_2)$$



Thm: 4

For each $C \in B$, $0 \leq P(C) \leq 1$

2m

proof:-

W.K.T ϕ is a proper subset of C.

ie., $\phi \subset C \subset \xi$

By thm: 3, $P(\phi) \leq P(C) \leq P(\xi)$

By thm: 2, and defn of probability set fn, we get

$$0 \le P(C) \le 1$$

Thm: 5 additional theorem of probability

If C_1 and C_2 are subsets of $\mathcal{C}$. then

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

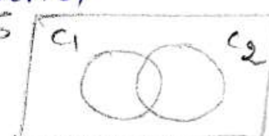
Proof:- We have $C_1 \cup C_2 = C_1 \cup (C_1^* \cap C_2)$ and

$C_1 \cap (C_1^* \cap C_2) = \phi$

$$\therefore P(C_1 \cup C_2) = P(C_1) + P(C_1^* \cap C_2) \rightarrow (1)$$

Also we have

$C_2 = (C_1 \cap C_2) \cup (C_1^* \cap C_2)$ and



$(C_1 \cap C_2) \cap (C_1^* \cap C_2) = \phi$

$$\therefore P(C_2) = P(C_1 \cap C_2) + P(C_1^* \cap C_2)$$

$$P(C_1^* \cap C_2) = P(C_2) - P(C_1 \cap C_2)$$

(1) becomes

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

Thm: 6

If C_1, C_2, C_3 are subsets of $\mathcal{C}$. Then

$$P(C_1 \cup C_2 \cup C_3) = P(C_1) + P(C_2) + P(C_3) - P(C_1 \cap C_2) - P(C_2 \cap C_3) - P(C_1 \cap C_3) + P(C_1 \cap C_2 \cap C_3)$$

Proof:-

$$P(C_1 \cup (C_2 \cup C_3)) = P(C_1) + P(C_2 \cup C_3) - P(C_1 \cap (C_2 \cup C_3))$$

$$= P(C_1) + P(C_2) + P(C_3) - P(C_2 \cap C_3) - P[(C_1 \cap C_2) \cup (C_1 \cap C_3)]$$

By Demorgan's law $P(A \cap (B \cup C)) = P[(A \cap B) \cup (A \cap C)]$

$$= P(C_1) + P(C_2) + P(C_3) - P(C_2 \cap C_3) - [P(C_1 \cap C_2) + P(C_1 \cap C_3) - P(C_1 \cap C_2 \cap C_3)]$$

$$= P(C_1) + P(C_2) + P(C_3) - P(C_1 \cap C_2) - P(C_2 \cap C_3) - P(C_1 \cap C_3) + P(C_1 \cap C_2 \cap C_3)$$

Thm: 7

If C_1 and C_2 are the subsets of the sample space $\mathcal{S}$, then $P(C_1 \cap C_2) \leq P(C_1 \cup C_2) \leq P(C_1) + P(C_2)$

Proof:-

We have $C_1 = (C_1 \cap C_2) \cup (C_1 \cap C_2^*)$ and $(C_1 \cap C_2) \cap (C_1 \cap C_2^*) = \phi$.

$$P(C_1) = P(C_1 \cap C_2) + P(C_1 \cap C_2^*)$$

$$P(C_1) - P(C_1 \cap C_2) = P(C_1 \cap C_2^*)$$

$$\geq 0$$

$C_2 = (C_1 \cap C_2) \cup (C_1^* \cap C_2)$

$$P(C_1) \geq P(C_1 \cap C_2)$$

$$P(C_1 \cap C_2) \leq P(C_1) \rightarrow (1)$$

Similarly, $P(C_1 \cap C_2) \leq P(C_2) \rightarrow (2)$

By the additional thm of prob,

$$P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

$$P(C_1 \cup C_2) - P(C_1) = P(C_2) - P(C_1 \cap C_2)$$

$$\geq 0 \quad [\text{By } (2)]$$

$$P(C_1) \leq P(C_1 \cup C_2) \rightarrow (3)$$

Now $P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$

$$P(C_1 \cap C_2) = P(C_1) + P(C_2) - P(C_1 \cup C_2)$$

$$P(C_1) + P(C_2) - P(C_1 \cup C_2) \geq 0 \quad [\because P(C_1 \cap C_2) \geq 0]$$

$$P(C_1) + P(C_2) \geq P(C_1 \cup C_2)$$

$$P(C_1 \cup C_2) \leq P(C_1) + P(C_2) \rightarrow (4)$$

From (1), (3) and (4), we get

$$P(C_1 \cap C_2) \leq P(C_1) \leq P(C_1 \cup C_2) \leq P(C_1) + P(C_2)$$

$$\therefore P(C_1 \cap C_2) \leq P(C_1 \cup C_2) \leq P(C_1) + P(C_2)$$

pblms:

1) Two coins are to be toasted. Let $c_1 = \{(H,H), (H,T)\}$ and $c_2 = \{(H,H), (T,H)\}$. Find $P(c_1 \cup c_2)$.

soln:- no of events = 4

$\mathcal{C} = \{(H,H), (H,T), (T,H), (T,T)\}$

$p(c_1) = 2/4$

$p(c_1) = 1/2, \quad p(c_2) = 1/2, \quad p(c_1 \cap c_2) = 1/4$

$p(c_1 \cup c_2) = 1/2 + 1/2 - 1/4 = 3/4$

2) The probability set fn assigns the probability of $1/36$ to each of the 36 pts ie., two dices are throw. Let $c_1 = \{(1,1), (2,1), (3,1), (4,1), (5,1)\}$, $c_2 = \{(1,2), (2,2), (3,2)\}$

$2^n = 2^6 = 36$

Find $p(c_1 \cup c_2)$.

soln:-

$\mathcal{C} = \{(1,1), (1,2), \ldots, (1,6), (2,1), \ldots, (2,6), \ldots, (6,6)\}$

$p(c_1) = 5/36, \quad p(c_2) = 3/36, \quad p(c_1 \cap c_2) = \phi$

$p(c_1 \cup c_2) = 2/9$.

Conditional probability

Defn:-

The conditional probability of the event c_2, given that c_1 provided that $p(c_1) > 0$ satisfies the following three conditions:

i) $p(c_2 | c_1) \geq 0$ (con prob of c_2 gvn c_1)

ii) $p(c_2 \cup c_3 \cup \ldots | c_1) = p(c_2|c_1) + p(c_3|c_1) + \ldots$ (| = gvn)

where $c_2, c_3, \ldots$ are mutually disjoint.

iii) $p(c_1 | c_1) = 1$.

Note:

The 3 conditions imply that the ~~problem~~ relation

$$P(C_2 \mid C_1) = \frac{P(C_1 \cap C_2)}{P(C_1)}$$

This is a suitable defn of the conditional probability of C_2 given C_1.

Law of total probability ques

5m 5m

2m (Let the space S be partition into k mutually exclusive and exhaustive events $C_1, C_2, \ldots, C_k \ni: P(C_i) > 0 \; i = 1, 2, \ldots, k$.

Here the events $C_1, C_2, \ldots, C_k$ do not need to be equally likely.

Let C be another event such that $P(C) > 0$

Thus C occurs with one and only one of the events $C_1, C_2, \ldots, C_k$.

i.e., $C = C \cap (C_1 \cup C_2 \cup \ldots \cup C_k)$ distributive law

$c_1 = (1,2)$ $\quad = (C \cap C_1) \cup (C \cap C_2) \cup \ldots \cup (C \cap C_k)$

$c_2 = (3,4)$

$c_3 = (5,6)$ Since $C \cap C_i$, $i = 1, 2, \ldots, k$ are mutually

$c = (2,5)$ exclusive, we have

$$P(C) = P(C \cap C_1) + P(C \cap C_2) + \ldots + P(C \cap C_k)$$

W.K.T the conditional probability is

$$P(C \mid C_i) = \frac{P(C \cap C_i)}{P(C_i)} \quad \forall \; i = 1, 2, \ldots, k$$

$$P(C \cap C_i) = P(C_i) \, P(C \mid C_i), \quad i = 1, 2, \ldots, k$$

$$\therefore P(C) = P(C_1) P(C \mid C_1) + P(C_2) P(C \mid C_2) + \ldots + P(C_k) P(C \mid C_k)$$

$$= \sum_{i=1}^{k} P(C_i) \, P(C \mid C_i)$$

2m.

This result is called the law of total probability.)

From the defn of conditional prob, and using the law of total prob, we have

$$p(c_j|c) = \frac{P(c \cap c_j)}{p(c)}$$

$$= \frac{p(c_j)\, p(c|c_j)}{\sum_{i=1}^{k} p(c_i)\, p(c|c_i)}$$

5m

This is known as Baye's theorem

Statement of Baye's thm:

2m If $c_1, c_2, \ldots, c_k$ are mutually exclusive and exhaustive events $p(c_i)>0$, $i=1,2,\ldots,k$. Let c be another event which is a subset of $\bigcup_{i=1}^{k} c_i$ such that $P(c)>0$ we have

$$p(c_j|c) = \frac{P(c_j)\, P(c|c_j)}{\sum_{i=1}^{k} P(c_i)\, P(c|c_i)}$$

Random variables:

consider a random experiment with the sample space $\mathcal{C}$. A function X which assigns to each element $c \in \mathcal{C}$ one and only real number $X(c) = x$ is called the random variable.

The space of X is the set of all real numbers

$$\mathcal{A} = \{x : x = X(c) : c \in \mathcal{C}\}$$

Induced probability:

A random variable is a function that carries the probability from a sample space $\mathcal{C}$ to a space of a real number with $A \subset \mathcal{A}$. The probability $P_X(A)$ is called the ~~the~~ induced probility.

Discrete set of points:

Let X denote the random variable with the one-dimensional space. Suppose that A consist of a countable no of points such a space A is called the discrete set of points.

Discrete random variable:

A random variable is said to be a discrete random variable if its space is either finite or countable.

Probability mass function (pmf)

Let X be a discrete random variable. Then the function $p(x)$ is said to be a pmf if $p(x)$ satisfies the following two conditions:

i) $0 \leq p(x) \leq 1$ ii) $\sum p(x) = 1$

Continuous random variable:-

A random variable X is said to be a continuous random variable if it can take any value in an interval which may be finite or infinite.

Probability density function: (pdf)

Let X be a continuous random variable. Then the function $f(x)$ is said to be a pdf if $f(x)$ satisfies the following two conditions:

i) $f(x) \geq 0$ ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Cumulative distribution function:

Let X be a random variable. Then the cumulative distribution function (c.d.f) of X or the distribution function of X is defined by

$$F_X(x) = P(X \le x), \quad -\infty < x < \infty$$

Note:

property of distribution function

$$P(a \le X \le b) = P(X \le b) - P(X \le a)$$
$$= F(b) - F(a)$$

Back face

Expectation of a random variable:

Let X be a random variable. Then the expectation of a random variable X is defined by (i) $E(X) = \sum x p(x)$ if X is a discrete random variable.

ii) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$, if X is a continuous random variable.

Sometimes $E(X)$ is called mathematical expectation of X or expected value of X or mean of X.

pblm:

1) Let the random variable x having the pdf given by the table

x	1	2	3	4
$f(x)$	$4/10$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

find $E(X)$

Soln:-

$$E(X) = \sum_x x f(x)$$
$$= \sum_1^4 x f(x)$$

$= 1 \times 4/10 + 2 \times 1/10 + 3 \times 3/10 + 4 \times 2/10$

$= 4/10 + 2/10 + 9/10 + 8/10$

$E(X) = \frac{23}{10}$

2) Let X having a p.d.f $f(x) = \begin{cases} 4x^3 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Find $E(X)$ ↓ mean

proof:-

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^1 x \cdot 4x^3\, dx$$

$$= 4\int_0^1 x^4\, dx$$

$$= 4\left[\frac{x^5}{5}\right]_0^1$$

$$= 4 \cdot 1/5 - 0$$

Mean $E(X) = 4/5$

3) For the probability density fn $f(x) = \begin{cases} k\,x(2-x) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

Find k and variance.

5m soln:-

To find k

Given $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\int_0^2 k\,x(2-x)\, dx = 1$$

$$k\int_0^2 (2x - x^2)\, dx = 1$$

$$k\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[x^2 - x^3/3\right]_0^2 = 1$$

$$K\left[4-\frac{8}{3}-0\right]=1$$

$$K\left[\frac{4}{3}\right]=1$$

$$K=\frac{3}{4}$$

To find variance:-

$$\sigma^2=E(X^2)-[E(X)]^2$$

$$E(x)=\int_{-\infty}^{\infty} x f(x)\,dx=1$$

$$=\int_0^2 x.\ Kx(2-x)\,dx$$

$$=K\int_0^2 x^2(2-x)\,dx$$

$$=K\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_0^2$$

$$=K\left[2.\frac{8}{3}-\frac{16}{4}\right]$$

$$=K\left[\frac{16}{3}-\frac{16}{4}\right]$$

$$=K\left[\frac{16}{3}-4\right]$$

$$=K\left[\frac{4}{3}\right]$$

$$=\frac{3}{4}\cdot\frac{4}{3}$$

$$E(X)=1$$

$$E(x^2)=\int_{-\infty}^{\infty} x^2.\ f(x)\,dx$$

$$=\int_0^2 x^2\ Kx(2-x)\,dx$$

$$=K\int_0^2 x^3(2-x)\,dx$$

$$=K\int_0^2 (2x^3-x^4)\,dx$$

$$=K\left[\frac{2x^4}{4}-\frac{x^5}{5}\right]_0^2$$

$$=K\left[\frac{2\times16}{4}-\frac{32}{5}\right]$$

$$= K\left[\frac{16}{2} - \frac{32}{5}\right]$$

$$= K\left[8 - \frac{32}{5}\right]$$

$$= K\left[\frac{40-32}{5}\right]$$

$$= K\left[8/5\right]$$

$$= 3/4 \cdot 8/5$$

$$= 6/5$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= 6/5 - 1$$

$$\sigma^2 = 1/5.$$

6) Let x have a p.d.f $f(x) = \begin{cases} \dfrac{x+2}{18} & -2<x<4 \\ 0 & \text{otherwise} \end{cases}$

Find i) $E(x)$ ii) $E(x+2)^3$ iii) $E(6x - 2(x+2)^3)$

Soln:-

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-2}^{4} x \cdot \left(\frac{x+2}{18}\right) dx$$

$$= \int_{-2}^{4} \frac{x^2+2x}{18}\, dx$$

$$= 1/18 \left[\frac{x^3}{3} + \frac{2x^2}{2}\right]_{-2}^{4}$$

$$= 1/18 \left[\frac{64}{3} + 16 - \left(-8/3 + 4\right)\right]$$

$$= 1/18 \left[\frac{64}{3} + 16 + 8/3 - 4\right]$$

$$= 1/18 \left[\frac{64+48+8-12}{3}\right]$$

$$= 1/18 \left[\frac{108}{3}\right]$$

$$= \frac{36}{18}$$

$$E(X) = 2$$

ii) $E(X+2)^3 = E(X^3) + E(6X^2) + 12E(X) + E(8)$

$$E(X^3) = \int_{-\infty}^{\infty} x^3 . f(x) . dx$$

$$= \int_{-2}^{4} x^3 \left(\frac{x+2}{18}\right) dx$$

$$= 1/18 \int_{-2}^{4} (x^4 + 2x^3) dx$$

$$= 1/18 \left[\frac{x^5}{5} + \frac{2x^4}{4}\right]_{-2}^{4}$$

$$= 1/18 \left[\frac{x^5}{5} + \frac{x^4}{2}\right]_{-2}^{4}$$

$$= 1/18 \left[\frac{1024}{5} + \frac{256}{2} - \left(\frac{-32}{5} + 16/2\right)\right]$$

$$= \frac{1}{18} \left[\frac{1024}{5} + 128 + \frac{32}{5} - 8\right]$$

$$= 1/18 \left[\frac{1024 + 640 + 32 - 40}{5}\right]$$

$$= 1/18 \left[\frac{1657}{5}\right]$$

$$= 1/18 \ (331.5)$$

$$= 18.4$$

$$E(X^2) = \int_{-2}^{4} x^2 \left(\frac{x+2}{18}\right) dx$$

$$= 1/18 \left[\frac{x^4}{4} + \frac{2x^3}{3}\right]_{-2}^{4}$$

$$= 1/18 \left[\frac{256}{4} + 2 . \frac{64}{3} - 16/4 + \frac{16}{3}\right]$$

$$= 1/18 \left[\frac{256-16}{4} + \frac{128+16}{3}\right]$$

$$= 1/18 \left[\frac{240}{4} + \frac{144}{3}\right]$$

$= \frac{1}{18}[60+48]$

$= \frac{1}{18}[108]$

$= 6.$

ii)

$= E(X^3) + E(6X^2) + 12E(X) + E(8)$

$= 18.4 + 6 \times 6 + 12 \times 2 + 8$

$= 86.4$

iii) $E(6X - 2(X+2)^3)$

$= E(6X) - E\,2(X+2)^3$

$= 6 \times 2 - 2 \times 86.4$

$= -160.8$

$= \frac{-804}{5}.$

pblms:

1) Show that the gn function $f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwis} \end{cases}$ is the pdf.

Soln:-

Cleary $f(x) > 0$

$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 2x\,dx = [x^2]_0^1 = 1$

$\therefore f(x)$ is a pdf.

2) $f(x) = e^{-x}$, $0 < x < \infty$. Find pdf

Soln:-

clearly $f(x) > 0$

$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} e^{-x}\,dx = \left(\frac{e^{-x}}{-1}\right)_0^{\infty}$

$= 1$

$f(x)$ is a pdf

3) Find the constant c so that satisfies the condition of pdf one variable $\ni$: $f(x)=\begin{cases} cxe^{-x} & 0<x<\infty \\ 0 & \text{otherwise} \end{cases}$

soln:-

$$\int_0^\infty c x e^{-x} dx = 1 \quad \int u dv = uv_1 - u'v_2$$

$$c[-xe^{-x} - e^{-x}]_0^\infty = 1$$

$$c = 1$$

4) Let $f(x) = \begin{cases} \frac{1}{x^2}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ $\ni$: $A_1 = \{x : 1 < x < 2\}$, $A_2 = \{x : 4 < x < 5\}$. Find (i) $P(A_1 \cup A_2)$ (ii) $P(A_1 \cap A_2)$

soln:-

$$A_1 \cap A_2 = \phi \Rightarrow P(A_1 \cap A_2) = 0$$

(i) $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

$$P(A_1) = \int_{A_1} f(x) dx = \int_1^2 \frac{1}{x^2} dx = \left[\frac{x^{-1}}{-1}\right]_1^2$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(A_2) = \int_4^5 \frac{1}{x^2} = \left[\frac{-1}{x}\right]_4^5 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

$$P(A_1 \cup A_2) = \frac{1}{2} + \frac{1}{20} = \frac{11}{20}$$

(ii) $P(A_1 \cap A_2) = 0$

5) S.T the function is gn by

x	0	1	2	3
$f(x)$	$1/10$	$3/10$	$2/10$	$4/10$

is pmf.

soln:- clearly $f(x) \geq 0$

$$\Sigma f(x) = \frac{1}{10} + \frac{3}{10} + \frac{2}{10} + \frac{4}{10} = 1$$

$\therefore$ $f(x)$ is pmf

Note:
The expected value of a product is not equal to the product of the expected values.

Example: 1
Let the R.V have the pdf

$$f(x)=\begin{cases} 1/5, & 0<x<5 \\ 0 & \text{otherwise} \end{cases}$$ Find (i) $E(5-X)$

(ii) $E[X(~~8~~5-X)]$

Soln:-
(i) $E(5-X) = 5 - E(X)$

W.K.T $E(X)=\int_{-\infty}^{\infty} x f(x)dx$

$$E(X)=\int_{0}^{5} x\cdot\frac{1}{5}dx$$

$$=\frac{1}{5}\left[\frac{x^2}{2}\right]_0^5$$

$$=\frac{1}{5}\left[\frac{25}{2}\right]$$

$$=\frac{5}{2}$$

$$E(5-X)= 5-\frac{5}{2}$$

$$=\frac{5}{2}$$

(ii) $E[x(5-x)] = \int_{-\infty}^{\infty} x(5-x)\, f(x)dx$

$$=\int_{0}^{5} x(5-x)\cdot\frac{1}{5}dx$$

$$=\frac{1}{5}\left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_0^5$$

$$=\frac{1}{5}\left[\frac{125}{2}-\frac{125}{3}\right]$$

$$=\frac{25}{6}$$

Note:
$$E(x)\cdot E(5-x) = \frac{5}{2}\cdot\frac{5}{2}$$

$$=\frac{25}{4}$$

$\therefore E[x(5-x)] \neq E(x)E(5-x)$

Homework

1) The RV having the pdf $f(x)=\frac{1}{2}(x+1), -1<x<1$

Find μ, σ^2.

Soln:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-1}^{1} x \frac{1}{2}(x+1)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^2+x)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{3}+\frac{1}{2}-\left(\frac{-1}{3}+\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}+\frac{1}{2}+\frac{1}{3}-\frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{2}{3}\right]$$

$$\mu = E(x) = \frac{1}{3}$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-1}^{1} x^2 \frac{1}{2}(x+1)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} (x^3+x^2)\,dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4}+\frac{x^3}{3}\right)_{-1}^{1}$$

$$= \frac{1}{2}\left(\frac{1}{4}+\frac{1}{3}-\left(\frac{1}{4}-\frac{1}{3}\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{4}+\frac{1}{3}-\frac{1}{4}+\frac{1}{3}\right)$$

$$= \frac{1}{2}\left(\frac{2}{3}\right)$$

$$E(x^2) = \frac{1}{3}$$

$$[E(X)]^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

$$\sigma^2 = \frac{1}{3} - \frac{1}{9}$$

$$= \frac{3-1}{9}$$

$$\sigma^2 = \frac{2}{9}$$

2) $f(x) = \frac{1}{5}$, $0 < x < 5$. Find μ, σ^2.

soln:-

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^5 \frac{1}{5}\, x\, dx$$

$$= \frac{1}{5}\left(\frac{x^2}{2}\right)_0^5$$

$$= \frac{1}{5}\left(\frac{25}{2}\right)$$

$$\mu = E(x) = \frac{5}{2}$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx$$

$$= \int_0^5 \frac{1}{5}\, x^2\, dx$$

$$= \frac{1}{5}\left(\frac{x^3}{3}\right)_0^5$$

$$= \frac{1}{5}\left(\frac{125}{3}\right)$$

$$E(x^2) = \frac{25}{3}$$

$$[E(x)]^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

$$\sigma^2 = \frac{25}{3} - \frac{25}{4}$$

$$= \frac{100-75}{12}$$

$$\sigma^2 = \frac{25}{12}$$

Mode:

Mode is the value of x for $f(x)$ is maximum i.e, $f'(x)=0$ and $f''(x)<0$.

pblm: 1 Find the mode of the dist $f(x)=\begin{cases} 12x^2(1-x), & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

soln:-

$$f(x) = 12x^2 - 12x^3$$

$$f'(x) = 24x - 36x^2$$

$$f''(x) = 24 - 72x$$

$$f'(x)=0 \Rightarrow \cancel{24}2x - 3x^2 = 0$$

$$x(2-3x)=0$$

$$x = 0, 2/3$$

$$f''(2/3) = 24 - 72(2/3)$$

$$= 24 - 48 < 0$$

$\therefore$ $f(x)$ is maximum

Mode $= 2/3$

2) Find the mode of the dist $f(x)=\frac{1}{2}x^2e^{-x}$, $0<x<\infty$

soln:-

$$f'(x) = xe^{-x} - \frac{1}{2}x^2e^{-x}$$

$$f''(x) = -2xe^{-x} + e^{-x} + \frac{1}{2}x^2e^{-x}$$

$$x=2 \Rightarrow f''(2) = -4e^{-2} + e^{-2} + \frac{1}{2}(4)e^{-2}$$

$$= e^{-2}(-4+1+2)$$

$$= -e^{-2}$$

$$f''(2) < 0$$

$\therefore$ $f(x)$ is maximum

Mode $= 2$.

Median

If M is median, then $\int_0^M f(x)\,dx = \frac{1}{2}$

1) Find the median of the distribution $f(x) = \begin{cases} 3x^2, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

Soln:-

$$\int_0^M 3x^2\,dx = \frac{1}{2} \Rightarrow M = \sqrt[3]{1/2}$$

$\left[\frac{3x^3}{3}\right]_0^M = 1/2$

$M^3 = 1/2$

2) Find the median of the dist $f(x) = \begin{cases} \frac{1}{\pi(1+x)^2}, & -\infty < x < \infty \\ 0, & \text{otherwise} \end{cases}$

Soln:-

$$\frac{1}{\pi}\int_0^M (1+x)^{-2}\,dx = \frac{1}{2}$$

$\int x^n = \frac{x^{n+1}}{n+1}$

$$\Rightarrow \frac{1}{\pi}\left[-(1+M)^{-1} + 1\right] = \frac{1}{2}$$

$$\Rightarrow \frac{-1}{1+M} + 1 = \frac{\pi}{2}$$

$$\Rightarrow \frac{M}{M+1} = \frac{\pi}{2}$$

$$\Rightarrow 2M - \pi M = \pi$$

$$\Rightarrow M = \frac{\pi}{2-\pi}$$

3) S.T the mean value of x does not exist for the pdf $f(x) = \begin{cases} \frac{1}{x^2}, & 1<x<\infty \\ 0, & \text{otherwise} \end{cases}$ Given

2m

Soln:-

$$E(x) = \int_1^\infty x \cdot \frac{1}{x^2}\,dx = \left[\log x\right]_1^\infty$$

$$= \infty$$

1) Given.
2) Formula
3) procedure

Hence the mean value of x does not exist.

Some special expectations

Moment generating function (MGF)

Let x be a r.v such that for some $h>0$, the expectation of e^{tX} exists for $-h<t<h$. The mgf of x is defined to be the function

$$M(t) = E(e^{tX}) \text{ for } -h < t < h$$

Note :-

(i) $M'(0) = E(X) = \mu$

(ii) $M''(0) = E(X^2)$

$\therefore \sigma^2 = M''(0) - [M'(0)]^2$

(iii) $M'''(0) = E(X^3)$

$\vdots$

In general, $M^{(n)}(0) = E(X^n)$ is called the n^{th} moment of X.

(iv) $M(t) = M(0) + \frac{t}{1!} M'(0) + \dots$

pblms

1) S.T the m.g.f of the r.v X having the pdf $f(x) = \begin{cases} 1/3, & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$ is

$$M(t) = \begin{cases} \frac{e^{2t} - e^{t}}{3t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

$\left[\frac{e^{tx}}{t}\right]_1^2$

soln:-

$$M(t) = E(e^{tX}) = \int_1^2 e^{tx} \cdot \frac{1}{3}\, dx = \frac{e^{2t} - e^{t}}{3t}, \ t \neq 0$$

At $t = 0$, $M(0) = E(1) = 1$.

$M(0) = E(e^{0 X}) = E(1)$

2) Find the m.g.f of $f(x) = \begin{cases} \frac{1}{k}, & x = 1, 2, \dots, k \\ 0, & \text{otherwise} \end{cases}$

discrete type

soln:-

$$M(t) = E(e^{tX}) = \sum_{x=1}^{k} e^{tx} \cdot \frac{1}{k} = \frac{e^t}{k}(1 + e^t + e^{2t} + \dots + e^{(k-1)t})$$

$$= \frac{e^t(1 - e^{kt})}{k(1 - e^t)}$$

$=\frac{1}{k}(e^t + e^{2t} + e^{3t} + \dots + e^{kt})$

n-1

3) Find the moments of the dist has the m.g.f $M(t) = (1-t)^{-3}$ for $t < 1$

soln:-

$$M'(t) = -3(1-t)^{-4}(-1)$$
$$= 3(1-t)^{-4}$$
$$M''(t) = 3(-4)(1-t)^{-5}(-1)$$
$$= 3.4(1-t)^{-5}$$
$$M'''(t) = 3.4.5(1-t)^{-6}$$
$$\vdots$$
$$M^{(n)}(t) = 3.4.5 \ldots (n+2)(1-t)^{-(n+3)}$$

At $t = 0$,

$$M'(0) = E(X) = 3$$
$$M''(0) = E(X^2) = 3.4$$
$$E(X^3) = 3.4.5$$
$$\vdots$$
$$E(X^n) = 3.4.5 \ldots (n+2)$$
$$= \frac{(n+2)!}{2}$$

4) Find the ~~first 3 moments~~ mean and variance for the pdf

$$f(x) = \begin{cases} \frac{3!}{x!(3-x)!}\left(\frac{1}{2}\right)^3, & x = 0,1,2,3 \\ 0, & \text{elsewhere.} \end{cases}$$

soln:-

when $x = 0$
$$f(x) = \frac{1}{8}$$

when $x=1$; $f(1) = \frac{3!}{1!\,2!}\left(\frac{1}{8}\right) = \frac{3}{8}$

when $x=2$; $f(2) = \frac{3!}{2!\,1!}\left(\frac{1}{8}\right) = \frac{3}{8}$

when $x=3$; $f(3) = \frac{3!}{3!\,0!}\left(\frac{1}{8}\right) = \frac{1}{8}$

$\therefore$ Given

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

W.k.T the m^{th} moment is $E(x^m)$

Also $E(x^m) = \sum x^m f(x)$

put $m=1$

$$E(x) = \sum x f(x)$$
$$= 0\times\frac{1}{8} + 1\times\frac{3}{8} + 2\times\frac{3}{8} + 3\times\frac{1}{8}$$
$$= \frac{12}{8}$$
$$= 3/2$$

$$E(x^2) = \sum x^2 f(x)$$
$$= 0 + \frac{3}{8} + \frac{12}{8} + \frac{9}{8}$$
$$= \frac{24}{8}$$
$$= 3$$

$$E(x^3) = \sum x^3 f(x)$$
$$= 0 + \frac{3}{8} + \frac{24}{8} + \frac{27}{8}$$
$$= \frac{54}{8}$$
$$= \frac{27}{4}.$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$
$$= 3 - (3/2)^2$$
$$= 3 - 9/4$$
$$= \frac{12-9}{4}$$
$$\sigma^2 = 3/4$$

5) Let $p(x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1, 2, \dots \\ 0 & \text{elsewhere} \end{cases}$ be the pmf of the R.V X. Find the mgf, mean and variance of X

Soln:- W.k.T the m.g.f is

$$M(t) = E[e^{tX}]$$
$$M(t) = \sum_{x=1}^{\infty} e^{tx} f(x)$$
$$= \sum e^{tx}\left(\frac{1}{2}\right)^x = \sum_{x=1,\dots}\left(\frac{1}{2}e^t\right)^x$$

$$= \frac{1}{2}e^t + \left(\frac{1}{2}e^t\right)^2 + \dots$$

$$= \frac{1}{2}e^t \left(1 + \frac{1}{2}e^t + \left(\frac{1}{2}e^t\right)^2 + \dots\right)$$

$$= \frac{1}{2}e^t \left(1 - \frac{1}{2}e^t\right)^{-1} \quad [\because (1-x)^{-1} = 1 + x + \dots]$$

$$= \frac{1}{2}e^t \left(\frac{2-e^t}{2}\right)^{-1}$$

$$M(t) = \frac{e^t}{2-e^t}$$

$$M'(t) = \frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2} = \frac{2e^t}{(2-e^t)^2}$$

$$M''(t) = \frac{(2-e^t)^2 \cdot 2e^t - 2e^t(2)(2-e^t)^1(-e^t)}{(2-e^t)^4}$$

At $t=0$,

$$M'(0) = E(X) = \frac{2}{(2-1)^2} = 2$$

$$M''(0) = E(X^2) = \frac{2(1) - 2(2)(1)(-1)}{1} = 6$$

Mean $= 2$

$$\text{Variance} = E(X^2) - [E(X)]^2$$
$$= 6 - 2^2$$
$$= 2.$$

6) Let the random variable have a mean μ and standard deviation σ and m.g.f $M(t)$, $-h < t < h$ show that

i) $E\left(\frac{X-\mu}{\sigma}\right) = 0$

ii) $E\left(\frac{X-\mu}{\sigma}\right)^2 = 1$

iii) $E\left[e^{t\left(\frac{X-\mu}{\sigma}\right)}\right] = e^{-\mu t/\sigma} M\left(t/\sigma\right)$

proof:-

i) $E\left(\frac{X-\mu}{\sigma}\right) = 0$

$$= \frac{1}{\sigma} E(x-\mu)$$

$$= \frac{1}{\sigma}\left[E(x) - E(\mu)\right]$$

$$= \frac{1}{\sigma}(\mu - \mu)$$

$$= 0$$

To prove

ii) $E\left(\frac{x-\mu}{\sigma}\right)^2 = 1$

$$E\left(\frac{x-\mu}{\sigma}\right)^2 = \frac{1}{\sigma^2}\left[E(x-\mu)^2\right]$$

$$= \frac{1}{\sigma^2}\left[E(x^2+\mu^2-2x\mu)\right]$$

$$= \frac{1}{\sigma^2}\left[E(x^2)+E(\mu^2) - 2E(x\mu)\right] \quad 2\mu E(X)$$

$$= \frac{1}{\sigma^2}\left[E(x^2)+\mu^2 - 2\mu^2\right]$$

$$= \frac{1}{\sigma^2}\left[E(x^2) - \mu^2\right]$$

$$= \frac{1}{\sigma^2}\left[E(x^2) - [E(x)]^2\right]$$

$$= \frac{1}{\sigma^2}\cdot\sigma^2$$

$$= 1$$

iii) To prove $E\left[e^{t\left(\frac{x-\mu}{\sigma}\right)}\right] = e^{-\mu t/\sigma} M\left(\frac{t}{\sigma}\right)$

2m

$$E\left[e^{t\left(\frac{x-\mu}{\sigma}\right)}\right] = E\left(e^{tx/\sigma}\cdot e^{-t\mu/\sigma}\right)$$

$$= e^{-t\mu/\sigma} E\left(e^{tx/\sigma}\right)$$

$$= e^{-t\mu/\sigma}\cdot M(t/\sigma) \quad \left[\because M(t) = E e^{tx}\right]$$

Homework

If $f(x) = \begin{cases} \frac{2}{x^3}, & 1 < x < \infty \\ 0 & \text{otherwise} \end{cases}$. Find mean and variance.

Soln:-

Given $f(x) = \begin{cases} \frac{2}{x^3}, & 1 < x < \infty \\ 0, & \text{otherwise} \end{cases}$

Mean $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_{1}^{\infty} x \cdot \frac{2}{x^3}\,dx$

$= \int_{1}^{\infty} \frac{2}{x^2}\,dx$

$= 2\left(\frac{x^{-1}}{-1}\right)_1^{\infty}$

$= 2\left(\frac{-1}{x}\right)_1^{\infty}$

$= 2(0+1)$

Mean $= 2$

Variance $\sigma^2 = E(x^2) - [E(x)]^2$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_{1}^{\infty} x^2 \frac{2}{x^3}\,dx$

$= \int_{1}^{\infty} \frac{2}{x}\,dx$

$= 2[\log x]_1^{\infty}$

$= \infty$

~~Hence the variance does not exist~~

$\sigma^2 = \infty - (2)^2$

$= \infty - 4$

$\sigma^2 = \infty$

Hence the variance does not exist

7) For the cauchy pdf $f(x) = \frac{1}{\pi(1+x^2)}$, $-\infty < x < \infty$, does the mgf exists?
cauchy pdf

V.Q 5m

Soln:-

Let x be a continuous R.v with pdf

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

Let $t > 0$ be given

If $x > 0$, then we have

$$\frac{e^{tx} - 1}{tx} = e^{\varepsilon_0} \geq 1 \text{ for some } 0 < \varepsilon_0 < tx$$

Hence $e^{tx} \geq 1 + tx \geq tx$.

W.k.T $M(t) = E(e^{tX})$

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\pi(1+x^2)} dx \geq \int_0^{\infty} e^{tx} \frac{1}{\pi(1+x^2)} dx$$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots$

$e^x \geq 1 + \frac{x}{1!}$

$e^{tx} \geq 1 + tx \geq tx$

$$\geq \int_0^{\infty} \frac{1}{\pi} \cdot \frac{tx}{1+x^2} dx$$

$e^{tx} \geq 1 + \frac{tx}{1!}$

$e^{tx} \geq \frac{tx}{1!}$

$$= \frac{t}{2\pi} \int_0^{\infty} \frac{d(1+x^2)}{1+x^2}$$

$$= \frac{t}{2\pi} \left[\log(1+x^2)\right]_0^{\infty}$$

$\int_0^{\infty} \frac{dx}{x} = \log x$

$$= \infty$$

$\therefore$ The integral does not exist.

Hence the mgf of the cauchy distribution does not exist.

8) Let x have a pdf $f(x) = \begin{cases} +ve, & x = -1, 0, 1 \\ 0, & \text{otherwise} \end{cases}$

i) If $f(0) = \frac{1}{2}$ find $E(x^2)$

ii) If $f(0) = \frac{1}{2}$ and $E(X) = \frac{1}{6}$ find $f(1), f(-1)$

Soln:-

Since $f(x)$ is pdf, $\sum_{x=-1}^{1} f(x) = 1$

$$f(-1) + f(0) + f(1) = 1$$

$$f(-1) + f(1) = 1 - 1/2$$

$$f(-1) + f(1) = \frac{1}{2} \rightarrow (1)$$

i) $E(x^2) = \sum_{x=-1}^{1} x^2 f(x)$

$$= f(-1) + 0 + f(1)$$

$$= f(-1) + f(1)$$

$E(x^2) = 1/2$ by (1)

ii) Given $E(x) = 1/6$

$$\sum_{x=-1}^{1} x f(x) = \frac{1}{6}$$

$$(-1) f(-1) + 0 + f(1) = \frac{1}{6}$$

$$-f(-1) + f(1) = \frac{1}{6} \rightarrow (2)$$

Solving (1) and (2)

(1) + (2) $\Rightarrow 2 f(1) = \frac{4}{6}$

$$f(1) = \frac{1}{3}$$

Sub $f(1) = \frac{1}{3}$ in (1)

$$f(-1) = \frac{1}{2} - \frac{1}{3}$$

$$f(-1) = \frac{1}{6}$$

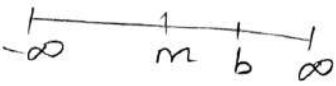
9) Let X be the R.V of continuous type that has a pdf $f(x)$. If m is unique median distribution of X and b is a real constant. Show that

U.Q 10m

$$E(|X-b|) = E(|X-m|) + 2\int_m^b (b-x) f(x)\,dx.$$

soln:-

$$E(|x-b|)=\int_{-\infty}^{\infty}|x-b|\,f(x)\,dx$$



$$=\int_{-\infty}^{b}|x-b|\,f(x)dx+\int_{b}^{\infty}|x-b|\,f(x)\,dx$$

$$=\int_{-\infty}^{b}(b-x)f(x)dx+\int_{b}^{\infty}(x-b)\,f(x)\,dx$$

$$=\int_{-\infty}^{m}(b-x)f(x)dx+\int_{m}^{b}(b-x)\,f(x)dx$$

$$+\int_{b}^{\infty}(x-b)\,f(x)dx \longrightarrow (1)$$

Now,

$$E(|x-m|)=\int_{-\infty}^{\infty}|x-m|\,f(x)\,dx$$

$$=\int_{-\infty}^{m}|x-m|\,f(x)dx+\int_{m}^{\infty}|x-m|\,f(x)\,dx$$

$$=\int_{-\infty}^{m}(m-x)\,f(x)dx+\int_{m}^{b}(x-m)f(x)dx$$

$$+\int_{b}^{\infty}(x-m)f(x)\,dx\rightarrow (2)$$

From (1) and (2)

$$E(|x-b|)-E(|x-m|)=\int_{-\infty}^{m}(b-x)\,f(x)\,dx$$

$$-\int_{-\infty}^{m}(m-x)\,f(x)\,dx+\int_{m}^{b}(b-x)\,f(x)\,dx$$

$$-\int_{m}^{b}(x-m)\,f(x)\,dx+\int_{b}^{\infty}(x-b)\,f(x)\,dx$$

$$-\int_{b}^{\infty}(x-m)\,f(x)\,dx.$$

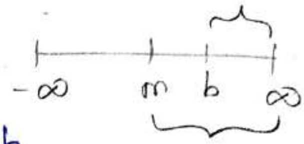
$$=\int_{-\infty}^{m}(b-x-m+x)\,f(x)dx+$$

$$\int_{m}^{b}(b-x-x+m)\,f(x)dx+$$

$$\int_{b}^{\infty}(x-b-x+m)\,f(x)\,dx.$$

$$= \int_{-\infty}^{m} (b-m) f(x)dx + \int_{m}^{b} (b-2x+m) f(x)dx + \int_{b}^{\infty} (m-b) f(x)dx.$$

$$= \int_{-\infty}^{m} (b-m) f(x)\,dx + \int_{m}^{b} (b-2x+m) f(x)dx + \int_{m}^{\infty} (m-b) f(x)dx - \int_{m}^{b} (m-b) f(x)dx.$$



$$= \int_{-\infty}^{m} (b-m) f(x)dx + \int_{m}^{b} (b-2x+m-m+b) f(x)dx + \int_{m}^{\infty} (m-b) f(x)dx.$$

$$= (b-m) \int_{-\infty}^{m} f(x)dx + (2b-2x) \int_{m}^{b} f(x)dx + (m-b) \int_{m}^{\infty} f(x)dx.$$

$$= (b-m)\frac{1}{2} + \int_{m}^{b} 2(b-x) f(x)dx + (m-b)\cdot\frac{1}{2}$$

$$= \frac{1}{2}(b-m+m-b) + 2\int_{m}^{b} (b-x) f(x)dx.$$

$$E(|x-b|) - E(|x-m|) = 2\int_{m}^{b} (b-x) f(x)dx.$$

$$\therefore E(|x-b|) = E(|x-m|) + 2\int_{m}^{b} (b-x) f(x)dx.$$

Relation between $f(x)$ and $F(x)$

Let x be a r.v having the pdf $f(x)$. Let $F(x)$ be the c.d.f of x. Then

$$\boxed{F'(x) = f(x)}$$

pblms

1) Find the mean of the dist for

$$F(x) = \begin{cases} 0, & x < 0 \\ x/8, & 0 \le x < 2 \\ x^2/16, & 2 \le x \le 4 \\ 0, & x > 4 \end{cases}$$

soln:

W.K.T $F'(x) = f(x)$

$$\therefore f(x) = \begin{cases} \frac{1}{8}, & 0 \le x \le 2 \\ x/8, & 2 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

$$E(x) = \int_0^2 x \cdot \frac{1}{8}\, dx + \int_2^4 x \cdot \frac{x}{8}\, dx$$

$$= \frac{1}{8}\left(\frac{x^2}{2}\right)_0^2 + \frac{1}{8}\left(\frac{x^3}{3}\right)_2^4$$

$$= \frac{1}{8}\left(\frac{4}{2}\right) + \frac{1}{8}\left(\frac{64}{3} - \frac{8}{3}\right)$$

$$= \frac{1}{4} + \frac{1}{8}\left(\frac{56}{3}\right)$$

$$= \frac{1}{4} + \frac{56}{24}$$

$$= \frac{6+56}{24}$$

$$= \frac{62}{24}$$

$$E(x) = \frac{31}{12}$$

2) Let x have the pmf $p(x) = \frac{1}{6}$, $i = 1, 2, \ldots 6$ Find cdf.

soln:-

x	1	2	3	4	5	6
$p(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

when $x < 1$, $F(x) = 0$

$1 \le x < 2$, $F(x) = 1/6$

$2 \le x < 3$, $F(x) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$

$3 \le x < 4$, $F(x) = 3/6$

$\vdots$

$6 \le x$, $F(x) = 1$.

3) pdf $f(x) = \begin{cases} \frac{2}{x^3}, & 1 < x < \infty \\ 0, & \text{otherwise} \end{cases}$ cdf?

soln:-

$$F(x) = \int_1^x \frac{2}{x^3}\, dx = 1 - \frac{1}{x^2}$$

$$F(x) = \begin{cases} 1 - \frac{1}{x^2}, & 1 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

4) If $f(x) = \begin{cases} \frac{x^2}{18}, & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$ Find $P(X^2 < 9)$

soln:-

$$P(X^2 < 9) = P(-3 < X < 3)$$

$$= \int_{-3}^{3} f(x)\, dx$$

$$= \frac{1}{18}\left[\frac{x^3}{3}\right]_{-3}^{3} = 1$$

5) The pdf of x in $p(x) = \begin{cases} \frac{x+2}{18}, & -2 < x < 3 \\ 0, & \text{otherwise} \end{cases}$

Find $P(X^2 < 9)$

soln:-

$$P(X^2 < 9) = P(-3 < X < 3)$$

$$= \int_{-3}^{-2} 0 + \int_{-2}^{3} f(x)\, dx$$

$$= \frac{1}{18}\left[\frac{x^2}{2} + 2x\right]_{-2}^{3}$$

$$= \frac{25}{36}$$

6) Let x have the cdf $F(x) = \begin{cases} 0, & x \le -1 \\ \frac{x+2}{4} & -1 < x < 1 \\ 1 & x \ge 1 \end{cases}$

Find i) $P(-1/2 < X < 1/2)$ ii) $P(X = 0)$ iii) $P(X = 1)$
iv) $P(-2 < X < 2)$ v) $P(2 < X < 3)$

soln:- i) $P\left(-\frac{1}{2} < X < \frac{1}{2}\right) = F\left(\frac{1}{2}\right) - F\left(\frac{-1}{2}\right)$

$$= \frac{\frac{1}{2}+2}{4} - \frac{-\frac{1}{2}+2}{4} = \frac{1}{4}$$

0-1 → left hand limit

ii) $P(X=0) = F(0) - F(0-) = \frac{2}{4} - 0 = \frac{1}{2}$

1-1 0

iii) $P(X=1) = F(1) - F(1-) = 1 - \frac{1+2}{4} = \frac{1}{4}$

iv) $P(-2 < X < 3) = F(3) - F(-2)$

$$= 1 - 0$$
$$= 1$$

v) $P(2 < X < 3) = F(3) - F(2)$

$$= 1 - 1$$
$$= 0$$

8) Let X be a r.v of the continuous type with pdf $f(x)$ which is positive provided $0 < x < b < \infty$ and is equal to zero elsewhere. Prove that $E(X) = \int_0^b [1 - F(x)]\,dx$ where $F(x)$ is the cdf of X.

soln:-

W.K.T $E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$E(X) = \int_0^b x\,f(x)\,dx$$

$F'(x) = f(x)$

$f(x) = \frac{d}{dx}[F(x)]$

$$= \int_0^b x \frac{d}{dx}[F(x)]\,dx$$

$$= \int_0^b x\,d[F(x)]$$

$\int u\,dv = uv - \int v\,du$

$u = x$ $dv = d[F(x)]$

$du = dx$ $v = F(x)$

$$= [x \cdot F(x)]_0^b - \int_0^b F(x)\,dx$$

$$= b\,F(b) - \int_0^b F(x)\,dx$$

put $F(b) = 1$

$$E(X) = b - \int_0^b F(x)\,dx$$

$$= \int_0^b dx - \int_0^b F(x)\,dx$$

$$= \int_0^b [1 - F(x)]\,dx.$$

7) If X has the cdf $F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & 0 \le x \end{cases}$

Find (i) the pdf (ii) $P(1 < X \le 3)$

Soln:-

i) $f(x) = F'(x) = \begin{cases} e^{-x}, & 0 \le x < \infty \\ 0, & \text{otherwise.} \end{cases}$

ii) $P(1 < X \le 3) = e^{-1} - e^{-3}$

$= 0.318$

$= F(3) - F(1)$

$= 1 - e^{-3} - 1 + e^{-1}$

$= e^{-1} - e^{-3}$

Extra problems

1) The distribution function of X is

$F(x) = \begin{cases} 0, & x < 1 \\ 1/6, & 1 \le x < 2 \\ 3/6, & 2 \le x < 3 \\ 1, & 3 \le x \end{cases}$ Find $P(1.5 < X \le 4.5)$

Soln:-

$P(1.5 < X \le 4.5) = F(4.5) - F(1.5)$

$= 1 - \frac{1}{6}$

$= \frac{5}{6}$

2) Let X be a discrete R.v. taking the values from $S = \{0, 1, 2, 3, 4\}$. Let the pdf be $f(x) = \frac{4!}{x!\,(4-x)!} \left(\frac{1}{2}\right)^4$. Find $P(A)$ if $A = \{0, 1\}$

Soln:-

$P(A) = \sum_{x=0}^{1} f(x)$

$= \frac{4!}{4!}\left(\frac{1}{2}\right)^4 + \frac{4!}{3!}\left(\frac{1}{2}\right)^4$

$= \frac{1}{16}(1+4)$

$= 5/16$

3) Let x be a discrete R.V taking the values of $S = \{1, 2, 3, \ldots\}$. Let $f(x) = \frac{1}{2^x}$ be the pdf. Find pr(A) where $A = \{1, 3, 5, 7, \ldots\}$

soln:- Given $f(x) = \frac{1}{2^x}$

$$P(A) = \sum_A f(x) = \sum_{x=1,3,\ldots} \frac{1}{2^x}$$

$$= \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \ldots$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2^2}\right]^{-1}$$

$$= \frac{1}{2} \times \frac{4}{3}$$

$$= 2/3.$$

4) Let the sample space $S = \{-\infty < c < \infty\}$ and $c \in S$ be defined by $\{c : 4 < c < \infty\}$. Let $p(c) = \int_c e^{-x} dx$. Evaluate i) $p(c)$ ii) $p(c^*)$ iii) $p(c \cup c^*)$

soln:-

i) $P(c) = \int_4^\infty e^{-x} dx = \left[-e^{-x}\right]_4^\infty = e^{-4}$

ii) $P(c^*) = 1 - P(c) = 1 - e^{-4}$

iii) $P(c \cup c^*) = P(c) + P(c^*) - P(c \cap c^*)$

$$= e^{-4} + 1 - e^{-4} - 0$$

$$= 1.$$

Markov's inequality

Let $U(X)$ be a non-negative function of the random variable X. If $E[U(X)]$ exists then for every positive constant c,

$$P[U(X) \geq c] \leq \frac{E[U(X)]}{c}$$

Proof:-

Let X be a continuous random variable with pdf $f(x)$

Let $U(X)$ be a non-~~empty~~ negative function of X.

Let c be the given constant.

Let $A = \{x : U(x) \geq c\}$ and $A^* = \{x \mid U(x) < c\}$, and $U(x) \geq 0$

Then $E[U(X)] = \int_{-\infty}^{\infty} U(x) f(x)\, dx$

$$= \int_A U(x) f(x)\, dx + \int_{A^*} U(x) f(x)\, dx \quad \text{——} (1)$$

Since $U(x) \geq 0$ and $f(x) \geq 0$, we have R.H.S of (1) is non-negative

$\therefore$ The L.H.S member is greater than or equal to either of there

ie., $E[U(X)] \geq \int_A U(x) f(x)\, dx$

If $x \in A$, then $U(x) \geq C$

$$E[U(X)] \geq \int_A c\, f(x)\, dx$$

$$= c\, P(X \in A)$$

$$= c\, P(U(X) \geq c)$$

$$P[U(X) \geq c] \leq \frac{E[U(X)]}{c}$$

Chebyshev's inequality :-

10m Let the random variable x have the distribution of probability about which we assume that there is a finite variance σ^2 and mean μ. Then every $k>0$, $P[|x-\mu| \geq k\sigma] \leq \frac{1}{k^2}$

Equivalently $P[|x-\mu| < k\sigma] \geq 1-\frac{1}{k^2}$

proof:- W.K.T Markov's inequality is

$$P[U(x) \geq c] \leq \frac{E[U(x)]}{c}$$

Take $U(x) = (x-\mu)^2$, $c = k^2\sigma^2$

$$P[(x-\mu)^2 \geq k^2\sigma^2] \leq \frac{E[(x-\mu)^2]}{k^2\sigma^2}$$

$$E[(x-\mu)^2] = E(x^2) + \mu^2 - 2\mu E(x)$$
$$= E(x^2) - [E(x)]^2$$
$$= \sigma^2$$

$$\therefore P[(x-\mu)^2 \geq k^2\sigma^2] \leq \frac{1}{k^2}$$

$$P[|x-\mu| \geq k\sigma] \leq \frac{1}{k^2}, \quad -P[|x-\mu| \geq k\sigma] \geq -1/k^2$$

Now $P[|x-\mu| < k\sigma] = 1 - P[|x-\mu| \geq k\sigma]$

$$\geq 1-\frac{1}{k^2}$$

Note:

$$k\sigma = \varepsilon \Rightarrow k = \frac{\varepsilon}{\sigma}$$

$$P[|x-\mu| \geq \varepsilon] \leq \frac{\sigma^2}{\varepsilon^2}$$

pblm:1)

repeated Let x be a r.v $\ni$: $E(x)=3$ and $E(x^2)=13$. Using chebyshev's inequality, determine a lower bound for $P[-2 \leq x < 8]$

proof:- $\mu = 3$, $\sigma^2 = 13-9 = 4$

$\sigma = 2$

$|x| < a$

$-a < x < a$

$$P[|x-3| < 2k] \geq 1 - \frac{1}{k^2}$$

$$P[-2k < x-3 < 2k] \geq 1 - \frac{1}{k^2}$$

$$P[3-2k < x < 3+2k] \geq 1 - \frac{1}{k^2} \rightarrow (1)$$

Here $3-2k = -2$ $\qquad$ $3+2k = 8$

$k = 5/2$ $\qquad$ $k = 5/2$

$\therefore k = 5/2$

$(1) \Rightarrow P[-2 < x < 8] \geq \frac{21}{25}$

2) The symmetric dice is thrown 600 times find a lower bound for the probability of 80 to 120 sixes.

repeated

Soln:-

Let x be the no of sixes

Let p = probability of getting six $= \frac{1}{6}$

$\therefore q = \frac{5}{6}$

Given $n = 600$

$\mu = np = 100$

$\sigma^2 = npq = \frac{250}{3}$

$\sigma = \frac{5\sqrt{10}}{\sqrt{3}}$

W.K.T Chebyshev's inequality is

$$P[|x-\mu| < k\sigma] \geq 1 - \frac{1}{k^2} \rightarrow (1)$$

$$P[|x-\mu| < k\sigma] = P\left[|x-100| < \frac{5\sqrt{10}}{\sqrt{3}}k\right]$$

$$= P\left[100 - \frac{5\sqrt{10}}{\sqrt{3}}k < x < 100 + \frac{5\sqrt{10}}{\sqrt{3}}k\right]$$

To find $P[80 < x < 120]$

$$100 - \frac{5\sqrt{10}}{\sqrt{3}}k = 80$$

$$k = \frac{20 \times \sqrt{3}}{5\sqrt{10}} = \frac{4\sqrt{3}}{\sqrt{10}}$$

$(1) \Rightarrow P(80 < X < 120) \geq 1 - \frac{10}{48}$

$= \frac{19}{24}.$

3) For a symmetric distribution $p(x) = 2^{-x}$ for $x = 1, 2, 3, \ldots$ prove that Chebyshev's inequality gives the probability $P[|x-\mu| \leq 2] \geq \frac{1}{2}$. Write the actual probability is $\frac{15}{16}$.

soln:-

Given $p(x) = 2^{-x}$, $x = 1, 2, 3, \ldots$

$\mu = E(X) = \sum_{x=1}^{\infty} x\, p(x)$

$= 1 \times 2^{-1} + 2 \times 2^{-2} + 3 \times 2^{-3} + \ldots$

$= \frac{1}{2}\left[1 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2^2} + \ldots\right]$

$= \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2}$ $\left[\because (1-x)^{-2} = 1 + 2x + 3x^2 + \ldots\right]$

$E(x) = 2$

$E(x^2) = \sum x^2 p(x)$

$= \sum (x^2 + x - x) p(x)$

$= \sum_{x=1}^{\infty} x(x+1) p(x) - \sum x p(x)$

$= \sum_{x=1}^{\infty} x(x+1) 2^{-x} - E(X)$

$= 1 \cdot 2 \cdot \frac{1}{2} + 2 \cdot 3 \cdot \frac{1}{2^2} + 3 \cdot 4 \cdot \frac{1}{2^3} + \ldots - 2$

$= \frac{1}{2}\left[1 \cdot 2 + 2 \cdot 3 \cdot \frac{1}{2} + 3 \cdot 4 \frac{1}{2^2} + \ldots\right] - 2$

$= \left(1 - \frac{1}{2}\right)^{-3} - 2$

$\left[\because (1-x)^{-3} = \frac{1}{1 \cdot 2}(1 \cdot 2 + 2 \cdot 3x + 3 \cdot 4x^2 + \ldots) = 1 + 3x + 6x^2 + \ldots\right]$

$= 8 - 2$

$= 6$

$\sigma^2 = E(x^2) - [E(x)]^2$

$= 6 - 4$

$= 2$

$\sigma = \sqrt{2}$

W.K.T Chebyshev's inequality is

$$P[|x-\mu| < k\sigma] \geq 1 - \frac{1}{k^2}$$

$K\sqrt{2} = 2$

$K = \frac{2}{\sqrt{2}}$

Gn $k\sigma = 2 \implies k = \sqrt{2}$

$$\therefore P[|x-2| \leq 2] \geq 1 - \frac{1}{2}$$

ie, $P[-2 < x-2 \leq 2] \geq \frac{1}{2}$

ie, $P[0 < x \leq 4] \geq \frac{1}{2}$

Now the actual probability is

$$P(0 < x \leq 4) = \sum_{x=1}^{4} p(x)$$

$$= \sum_{x=1}^{4} 2^{-x}$$

$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$$

$$= \frac{8+4+2+1}{16}$$

$$= \frac{15}{16}.$$

4) Let x have the pdf $f(x) = \begin{cases} \frac{1}{2\sqrt{3}}, & -\sqrt{3} < x < \sqrt{3} \\ 0, & \text{otherwise} \end{cases}$

If $k = \frac{3}{2}$ Find the upper bound and check it.

Soln:- First to find : μ, σ^2

Given $f(x) = \begin{cases} 1/2\sqrt{3}, & -\sqrt{3} < x < \sqrt{3} \\ 0, & \text{otherwise} \end{cases}$

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} x \frac{1}{2\sqrt{3}}\, dx$$

$$= \frac{1}{2\sqrt{3}} \left(\frac{x^2}{2}\right)_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}} \left[\frac{3}{2} - \frac{3}{2} \right]$$

$$E(x) = \mu = 0$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} x^2 \frac{1}{2\sqrt{3}}\, dx$$

$$= \frac{1}{2\sqrt{3}} \left(\frac{x^3}{3}\right)_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}} \left(\frac{3\sqrt{3}}{3} + \frac{3\sqrt{3}}{3} \right)$$

$$= \frac{3}{3(2\sqrt{3})} (\sqrt{3} + \sqrt{3})$$

$$= \frac{2\sqrt{3}}{2\sqrt{3}}$$

$$E(x^2) = 1$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= 1 - 0$$

$$\sigma^2 = 1$$

$$\sigma = 1$$

Upper bound:

W.K.T $P[|X-\mu| \geq K\sigma] \leq \frac{1}{K^2}$

ie, $P[|x| \geq \frac{3}{2}] \leq \frac{4}{9}$

$\therefore$ Upper bound $= \frac{4}{9}$

Actual probability is

$$P[|x| \geq \tfrac{3}{2}] = 1 - P[|x| < 3/2]$$

$$= 1 - P[-3/2 < x < 3/2]$$

$$= 1 - \int_{-3/2}^{3/2} f(x)\,dx$$

$$= 1 - \int_{-3/2}^{3/2} \frac{1}{2\sqrt{3}}\,dx$$

$$= 1 - \frac{1}{2\sqrt{3}} \left[x\right]_{-3/2}^{3/2}$$

$$= 1 - \frac{1}{2\sqrt{3}} \times 3$$

$$= 1 - \frac{\sqrt{3}}{2} \leq \frac{4}{9}$$

UNIT - II

MULTIVARIATE DISTRIBUTIONS

Unit - II

Multivariate distributions

Distribution of two random variables

Defn:-

Given a random experiment with a sample space $\mathcal{S}$. consider two random variables X_1, X_2 which assigns to each element c of $\mathcal{S}$ one and only one ordered pair of numbers $X_1(c) = x_1$ and $X_2(c) = x_2$. Then we say that (X_1, X_2) is a random vector.

The space of (X_1, X_2) is the set of all ordered pairs $\mathcal{D} = \{(x_1, x_2) : x_1 = X_1(c), x_2 = X_2(c), c \in \mathcal{S}\}$

Note:

X_1, X_2 is a vector function from $\mathcal{S}$ into $\mathcal{D}$

Defn: probability

Let $\mathcal{D}$ be the space associated with the random vector (X_1, X_2). Let A be a subset of $\mathcal{D}$. Then the probability of the event A is defined by

$$P(A) = Pr[(x_1, x_2) \in A] = \sum_A \sum f(x_1, x_2) \text{ (discrete)}$$

$$= \iint_A f(x_1, x_2)\, dx_1\, dx_2 \text{ (continuous)}$$

In each case $f(x_1, x_2)$ is called the joint p.d.f of two variables x_1 and x_2.

pblm

1) Let $f(x,y) = \begin{cases} 6x^2y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$ be the pdf of two random variables X and Y.

Find $Pr(0 < X < \frac{3}{4}, \frac{1}{3} < Y < 2)$.

soln:-

W.K.T $Pr[(x_1, x_2) \in A] = \iint_A f(x_1, x_2)\, dx_1\, dx_2$

$$P(0 < X < 3/4, \tfrac{1}{3} < Y < 2) = \int_{1/3}^{2} \int_0^{3/4} f(x,y)\, dx\, dy$$

$$= \int_{1/3}^{1} \int_0^{3/4} 6x^2y\, dx\, dy$$

$$= 6 \int_{1/3}^{1} y \cdot \left[\frac{x^3}{3}\right]_0^{3/4} dy$$

$$= 6 \int_{1/3}^{1} y \left[\frac{27}{64 \times 3}\right] dy$$

$$= 6 \left[\frac{9}{64}\right] \left[\frac{y^2}{2}\right]_{1/3}^{1}$$

$$= \frac{27}{64}\left(1 - \frac{1}{9}\right)$$

$$= \frac{27}{64} \times \frac{8}{9}$$

$$= \frac{3}{8}$$

2) Let $f(x_1, x_2) = \begin{cases} 4x_1 x_2, & 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$ be a pdf of X_1 and X_2. Find $P(0 < X_1 < 1/2, \frac{1}{4} < X_2 <$

soln:-

– W.K.T $P[(x_1, x_2) \in A] = \iint_A f(x_1, x_2)\, dx_1\, dx_2$

$$P(0 < X_1 < 1/2, \tfrac{1}{4} < X_2 < 1) = \cancel{\int_{1/4}^{1} \int_{0}^{1/2} f(x,y)\, dx\, dy}$$

$$= \int_{1/4}^{1} \int_{0}^{1/2} f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{1/4}^{1} \int_{0}^{1/2} 4x_1 x_2\, dx_1\, dx_2$$

$$= 4\int_{1/4}^{1} x_2 \left[\frac{x_1^2}{2}\right]_0^{1/2} dx_2$$

$$= 4\int_{1/4}^{1} x_2\, \frac{1}{8}\, dx_2$$

$$= \frac{1}{2} \int_{1/4}^{1} x_2\, dx_2$$

$$= \frac{1}{2} \left[\frac{x_2^2}{2}\right]_{1/4}^{1}$$

$$= \frac{1}{2} \left[-\frac{1/16}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{4} \left[-\frac{1}{16} + 1\right]$$

$$= \frac{1}{4} \left[\frac{-1+16}{16}\right]$$

$$P(0 < X_1 < 1/2, \tfrac{1}{4} < X_2 < 1) = \frac{15}{64}$$

Distribution function

Let the random variables X and Y have the probility set function $P(A)$ where A is a two dimensional set. Then the distribution function of X and Y is defined by

$$F(x,y) = P(X \le x, Y \le y)$$

In discrete type,

$$F(x,y) = \sum_{-\infty < x < x} \sum_{-\infty < y < y} f(x,y)$$

In continuous type,

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)\, dx\, dy.$$

Marginal pdf:

The marginal pdf of X_1 is defined by

$$f_{X_1}(x_1) = f_1(x_1) = f(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2 \quad \text{(cts)}$$

$$= \sum_{x_2} f(x_1, x_2) \quad \text{(discrete)}$$

The marginal pdf of X_2 is defined by

$$f_{X_2}(x_2) = f_2(x_2) = f(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1 \quad \text{(cts)}$$

$$= \sum_{x_1} f(x_1, x_2) \quad \text{(discrete)}$$

Conditional distribution

Let X_1 and X_2 denote random variables which have the joint pdf $f(x_1, x_2)$. Let $f_1(x_1)$ and $f_2(x_2)$ denote the marginal pdf of X_1 and X_2 respectively.

We define the symbol

$$f_{2|1}(x_2|x_1) = \frac{f(x_1, x_2)}{f(x_1)} \begin{array}{l} \to \text{joint pdf} \\ \to \text{marginal pdf} \end{array}$$

It is called the conditional pdf of the random variable x_2, given that the random variable $X_1 = x_1$

Similarly, we define the symbol

$$f_{1|2}(x_1|x_2) = \frac{f(x_1, x_2)}{f(x_2)}$$

It is called the conditional pdf of the random variable x_1, given that the random variable $X_2 = x_2$.

Conditional expectation:

Let $u(x_1)$ be a function of x_1, then the conditional expectation of $u(x_1)$ given that $X_2 = x_2$ is defined by

2m

$$E[u(x_1) | X_2 = x_2] = \int_{-\infty}^{\infty} u(x_1) f(x_1|x_2) dx_1 \text{ (cts)}$$

conditional mean of X_1 given $X_2 = x_2$ is

$$E[X_1 | X_2 = x_2]$$

conditional variance of X_1 given $X_2 = x_2$ is

$$\operatorname{var}(X_1|X_2 = x_2) = E(X_1^2|X_2 = x_2) - [E(X_1|X_2 = x_2)]^2$$

pblms:

1) Let X_1 and X_2 have the joint pdf

repeated 5m

$$f(x_1, x_2) = \begin{cases} x_1 + x_2, & 0 < x_1 < 1, \ 0 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the marginal pdf of X_1 and X_2.

Soln:-

The marginal pdf of X_1 is

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_2 .$$

$$= \int_0^1 (x_1 + x_2)\, dx_2$$

$$= \left[x_1 x_2 + \frac{x_2^2}{2} \right]_0^1$$

$$= x_1 + \frac{1}{2}$$

$$\therefore f_1(x_1) = \begin{cases} x_1 + \frac{1}{2}, & 0 < x_1 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

The marginal pdf of X_2 is

$$f_2(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$$

$$= \int_0^1 (x_1 + x_2)\, dx_1$$

$$= \left[\frac{x_1^2}{2} + x_2 x_1 \right]_0^1$$

$$= \frac{1}{2} + x_2$$

$$\therefore f_2(x_2) = \begin{cases} \frac{1}{2} + x_2, & 0 < x_2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

2) Let X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = \begin{cases} 2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$$ Find

i) Marginal pdf ii) conditional pdf

iii) conditional mean and conditional variance of X_1 given $X_2 = x_2$.

iv) $P(0 < X_1 < \frac{1}{2})$ (v) $P(0 < X_1 < \frac{1}{2} \mid X_2 = \frac{3}{4})$

soln:-

i) Marginal pdf of X_1 is

$$f(x_1) = \int_{x_1}^{1} 2\, dx_2$$

$$= 2(1 - x_1)$$

$$f(x_1) = \begin{cases} 2(1-x_1), & 0 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}$$

Marginal pdf of X_2 is

$$f(x_2) = \int_0^{x_2} 2\,dx_1$$

$$= 2x_2$$

$$f(x_2) = \begin{cases} 2x_2, & 0 < x_2 < 1 \\ 0, & \text{elsewhere} \end{cases}$$

ii) The conditional pdf of X_1 given $X_2 = x_2$ is

$$f(x_1 | x_2) = \frac{f(x_1, x_2)}{f(x_2)}$$

$$= \frac{2}{2x_2} = \frac{1}{x_2}$$

$$\therefore f(x_1 | x_2) = \begin{cases} \frac{1}{x_2}, & 0 < x_1 < x_2 \\ 0, & \text{otherwise} \end{cases}$$

The conditional pdf of X_2 given $X_1 = x_1$ is

$$f(x_2 | x_1) = \frac{f(x_1, x_2)}{f(x_1)}$$

$$= \frac{2}{8(1-x_1)} = \frac{1}{1-x_1}$$

$$\therefore f(x_2 | x_1) = \begin{cases} \frac{1}{1-x_1}, & x_1 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

iii) conditional mean of X_1 given $X_2 = x_2$ is

$$E[X_1 | X_2 = x_2]$$

W.K.T $E[X_1 | x_2] = \int_{-\infty}^{\infty} x_1 f(x_1 | x_2)\,dx_1$

$$= \int_0^{x_2} x_1 \frac{1}{x_2}\,dx_1$$

$$= \frac{1}{x_2}\left[\frac{x_1^2}{2}\right]_0^{x_2}$$

$$E(X_1 | X_2 = x_2) = \frac{x_2}{2}$$

Conditional variance of X_1 given $X_2 = x_2$ is

$$E[X_1^2 | X_2 = x_2] - [E(X_1 | X_2 = x_2)]^2$$

$$E(x_1^2 \mid x_2) = \int_{-\infty}^{\infty} x_1^2 f(x_1 \mid x_2)\, dx_1$$

$$= \int_0^{x_2} x_1^2 \cdot \frac{1}{x_2}\, dx_1$$

$$= \frac{1}{x_2}\left[\frac{x_1^3}{3}\right]_0^{x_2}$$

$$= \frac{x_2^2}{3}$$

$$\therefore \operatorname{var}(X_1 \mid X_2 = x_2) = \frac{x_2^2}{3} - \left(\frac{x_2}{2}\right)^2$$

$$= x_2^2\left(\frac{4-3}{12}\right)$$

$$= \frac{x_2^2}{12}$$

iv) $$P(0 < X_1 < 1/2) = \int_0^{1/2} f(x_1)\, dx_1$$

$$= \int_0^{1/2} 2(1-x_1)\, dx_1$$

$$= 2\left(x_1 - \frac{x_1^2}{2}\right)_0^{1/2}$$

$$= 2\left[\frac{1}{2} - \frac{1}{8}\right]$$

$$= 2\left(\frac{3}{8}\right)$$

$$= \frac{3}{4}$$

v) $$P\left(0 < X_1 < \frac{1}{2} \mid X_2 = \frac{3}{4}\right) = \int_0^{1/2} f\left(x_1 \mid x_2 = \frac{3}{4}\right) dx_1$$

$$= \int_0^{1/2} \left[\frac{1}{x_2}\right]_{x_2 = 3/4} dx_1$$

$$= \frac{4}{3}\left(x_1\right)_0^{1/2}$$

$$= \frac{2}{3}.$$

3) Let X_1 and X_2 have the pdf

$$f(x_1, x_2) = \begin{cases} 8x_1 x_2 , & 0 < x_1 < x_2 < 1 \\ 0 , & \text{otherwise} \end{cases}$$

Find i) $E(X_1 X_2^2)$ ii) $E(X_2)$

iii) $E(7X_1 X_2^2 + 5X_2)$.

soln:-

i) $$E(X_1 X_2^2) = \int_0^1 \int_0^{x_2} x_1 x_2^2 f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^1 \int_0^{x_2} x_1 x_2^2 \cdot 8x_1 x_2\, dx_1\, dx_2$$

$$= 8 \int_0^1 x_2^3 \left[\frac{x_1^3}{3}\right]_0^{x_2} dx_2$$

$$= \frac{8}{3}\left(\frac{x_2^7}{7}\right)_0^1$$

$$= \frac{8}{21}$$

ii) $$E(X_2) = \int_0^1 \int_0^{x_2} x_2 f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^1 \int_0^{x_2} x_2 \cdot 8x_1 x_2\, dx_1\, dx_2$$

$$= 8 \int_0^1 x_2^2 \left(\frac{x_1^2}{2}\right)_0^{x_2} dx_2$$

$$= 4\left(\frac{x_2^5}{5}\right)_0^1$$

$$= 4/5.$$

iii) $$E(7X_1 X_2^2 + 5X_2) = 7E(X_1 X_2^2) + 5E(X_2)$$

$$= 7\left(\frac{8}{21}\right) + 5\left(\frac{4}{5}\right)$$

$$= \frac{8}{3} + 4$$

$$= \frac{20}{3}.$$

4) Let X and Y have joint pdf $f(x,y) = \begin{cases} \frac{1}{3}, (x,y) = (0,0), (0,1), (1,1) \\ 0, \text{otherwise} \end{cases}$

Find $E\left[(X - \frac{1}{3})(Y - \frac{2}{3})\right]$

soln:

y \ x	0	1	$f_Y(y)$
0	$\frac{1}{3}$	0	$\frac{1}{3}$
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
$f_X(x)$	$\frac{2}{3}$	$\frac{1}{3}$	

Marginal pdf of X is

x	0	1
$f(x)$	$2/3$	$1/3$

The marginal pdf of Y is

y	0	1
$f(y)$	$\frac{1}{3}$	$\frac{2}{3}$

$$E\left[(X - \tfrac{1}{3})(Y - \tfrac{2}{3})\right] = E\left[XY - \tfrac{2}{3}X - \tfrac{1}{3}Y + \tfrac{2}{9}\right]$$

$$= E(XY) - \tfrac{2}{3}E(X) - \tfrac{1}{3}E(Y) + \tfrac{2}{9}$$

$$= \sum_y \sum_x xy\, f(x,y) - \tfrac{2}{3}\sum_x x f(x) - \tfrac{1}{3}\sum_y y f(y) + \tfrac{2}{9}$$

$$= \tfrac{1}{3} - \tfrac{2}{3}\left(\tfrac{1}{3}\right) - \tfrac{1}{3}\left(\tfrac{2}{3}\right) + \tfrac{2}{9}$$

$$= \tfrac{1}{3} - \tfrac{2}{9}$$

$$= \tfrac{1}{9}.$$

5) Let $F(x,y)$ be the distribution function of X and Y S.T

$$P(a \le X \le b, c \le Y \le d) = F(b,d) - F(b,c) - F(a,d) + F(a,c)$$

Soln:-

$$P(a \le X \le b, c \le Y \le d) = P(a \le X \le b, Y \le d) - P(a \le X \le b, Y \le c)$$

$$= P(X \le b, Y \le d) - P(X \le a, Y \le d) - [P(X \le b, Y \le c) - P(X \le a, Y \le c)]$$

$$[\because P(a \le X \le b) = P(X \le b) - P(X \le a)]$$

$$= F(b,d) - F(a,d) - F(b,c) + F(a,c)$$

6) Let X_1 and X_2 be the random variables of continuous type that have the joint pdf $f(x_1, x_2)$ and the marginal pdf $f(x_1)$ and $f(x_2)$ respectively, then P.T

Theorem:

i) $E[E(X_2|X_1)] = E(X_2)$

5m

ii) $Var(X_2) \ge Var[E(X_2|X_1)]$

Soln:-

i) W.K.T $E(X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 f(x_1, x_2) dx_1 dx_2$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 f(x_1, x_2) dx_2 dx_1$$

[By change the order of $\int$]

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 \frac{f(x_1, x_2)}{f(x_1)} f(x_1) dx_2 dx_1$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_2 f(x_2|x_1) dx_2\right] f(x_1) dx_1$$

$$= \int_{-\infty}^{\infty} E(X_2|x_1) f(x_1) dx_1$$

$$= E[E(X_2|X_1)]$$

ii) Let $\mu_2 = E(X_2)$

Formula: $var(X) = E(X-\mu)^2$

$$var(X_2) = E(X_2 - \mu_2)^2$$

$$= E[(X_2 - E(X_2|X_1)) + (E(X_2|X_1) - \mu_2)]^2$$

$$= E[X_2 - E(X_2|X_1)]^2 + E[E(X_2|X_1) - \mu_2]^2 + 2E\{[X_2 - E(X_2|X_1)][E(X_2|X_1) - \mu_2]\} \rightarrow (1)$$

$$2E\{[X_2 - E(X_2|X_1)][E(X_2|X_1) - \mu_2]\}$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_2 - E(X_2|x_1)][E(X_2|x_1) - \mu_2] f(x_1, x_2)\, dx_1\, dx_2$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_2 - E(X_2|x_1)][E(X_2|x_1) - \mu_2] \frac{f(x_1, x_2)}{f(x_1)} f(x_1)\, dx_2\, dx_1$$

[By change the order of $\int$n]

$$= 2\int_{-\infty}^{\infty}[E(X_2|x_1) - \mu_2]\left\{\int_{-\infty}^{\infty}[x_2 - E(X_2|x_1)] f(x_2|x_1)\, dx_2\right\} f(x_1)\, dx_1$$

But $E(X_2|X_1)$ is the conditional mean of X_2 given $X_1 = x_1$

Here $\int_{-\infty}^{\infty}[x_2 - E(X_2|x_1)] f(x_2|x_1)\, dx_2$

$$= E(X_2|x_1) - E(X_2|x_1)$$

$$= 0$$

$\therefore 2E\{[X_2 - E(X_2|X_1)][E(X_2|X_1) - \mu_2]\} = 0$

(1) becomes

$$var(X_2) = E[X_2 - E(X_2|X_1)]^2 + E[E(X_2|X_1) - \mu_2]^2$$

Here $E[X_2 - E(X_2|X_1)]^2 \geq 0$ and

$$E[E(X_2|X_1) - \mu_2]^2 \geq 0$$

$$\therefore \text{var}(X_2) \geq E[E(X_2|X_1) - \mu_2]^2$$
$$= \text{var}[E(X_2|X_1)]$$

$$\therefore \text{var}(X_2) \geq \text{var}[E(X_2|X_1)]$$

Correlation coefficient:

Let x and y have the joint pdf of (x,y). Let $u(x,y)$ be a function of x and y. The means of x and y say μ_1 and μ_2 are obtained by taking $u(x,y)$ to be x and y respectively. The variances of x and y say σ_1^2 and σ_2^2 are obtained by setting the function $u(x,y) = (x-\mu_1)^2$ and $(y-\mu_2)^2$ respectively.

Then

$$E[(X-\mu_1)(Y-\mu_2)] = E(XY) - \mu_2 E(X) - \mu_1 E(Y) + \mu_1\mu_2$$
$$= E(XY) - \mu_1\mu_2$$
$$= E(XY) - E(X)E(Y)$$

$$\therefore E[(X-\mu_1)(Y-\mu_2)] = E(XY) - E(X)E(Y)$$

This is called the co-variance of X & Y and it is denoted by $\text{cov}(X,Y)$ 2m defn.

The number $\rho = \dfrac{\text{cov}(X,Y)}{\sigma_1\sigma_2} = \dfrac{E(XY) - E(X)E(Y)}{\sigma_1\sigma_2}$

is called the correlation coefficient of X and Y.

Note:

(i) $E[\mathrm{var}(Y|x)] = \sigma_2^2(1-\rho^2)$

(ii) $E[\mathrm{var}(x|y)] = \sigma_1^2(1-\rho^2)$

(iii) $E(XY) = \rho\sigma_1\sigma_2 + \mu_1\mu_2$

Thm: If the correlation coefficient $\rho(X,Y)$ exist, then prove that $-1 \le \rho \le 1$

proof:-

W.K.T $E[\mathrm{var}(Y|x)] = \sigma_2^2(1-\rho^2)$

$$\sigma_2^2(1-\rho^2) \ge 0$$

$$\Rightarrow 1-\rho^2 \ge 0$$

$$\rho^2 \le 1$$

$$\therefore -1 \le \rho \le 1$$

pblms

1) Let the r.v X and Y have joint pdf

$$f(x,y) = \begin{cases} x+y & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

5m Find the correlation coefficient of X & Y.

soln:-

W.K.T correlation coefficient of X and Y is

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sigma_1\sigma_2}$$

$$E(X) = \int_0^1\int_0^1 x f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x(x+y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (x^2+xy)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + \frac{x^2y}{2}\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{1}{3} + \frac{y}{2}\right] dy$$

$$= \left[\frac{1}{3}y + \frac{1}{2}\frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

$$E(X^2) = \int_0^1 \int_0^1 x^2 f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 x^2 (x+y)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^4}{4} + y \cdot \frac{x^3}{3}\right]_0^1 dy$$

$$= \left[\frac{1}{4}y + \frac{1}{3} \cdot \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{6}$$

$$= \frac{5}{12}$$

$$\sigma_1^2 = E(x^2) - [E(x)]^2$$

$$= \frac{5}{12} - \frac{49}{144}$$

$$= \frac{11}{144}$$

$$E(Y) = \int_0^1 \int_0^1 y\, f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 y(x+y)\, dx\, dy$$

$$= \int_0^1 (xy + y^2)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2 y}{2} + y^2 \cdot x\right]_0^1 dy$$

$$= \int_0^1 \left(\frac{y}{2} + y^2\right) dy$$

$$= \left[\frac{1}{2} \frac{y^2}{2} + \frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{4} + \frac{1}{3}$$

$$= \frac{7}{12}$$

$$E(Y^2) = \int_0^1 \int_0^1 y^2(x+y)\, dx\, dy$$

$$= \int_0^1 \left[y^2 \frac{x^2}{2} + y^3 x\right]_0^1 dy$$

$$= \left[\frac{1}{2} \cdot \frac{y^3}{3} + \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{6} + \frac{1}{4}$$

$$= \frac{5}{12}$$

$$\sigma_2^2 = E(Y^2) - [E(Y)]^2$$
$$= \frac{5}{12} - \frac{49}{144}$$
$$= \frac{11}{144}$$

$$E(XY) = \int_0^1 \int_0^1 xy\, f(x,y)\, dx\, dy$$
$$= \int_0^1 \int_0^1 xy(x+y)\, dx\, dy$$
$$= \int_0^1 \int_0^1 (x^2 y + xy^2)\, dx\, dy$$
$$= \int_0^1 \left[\frac{x^3}{3} y + \frac{x^2}{2} \cdot y^2\right]_0^1 dy$$
$$= \left[\frac{1}{3} \cdot \frac{y^2}{2} + \frac{1}{2} \cdot \frac{y^3}{3}\right]_0^1$$
$$= \frac{1}{6} + \frac{1}{6}$$
$$= \frac{1}{3}$$

$$\therefore \rho = \frac{\frac{1}{3} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right)}{\frac{\sqrt{11}}{12} \cdot \frac{\sqrt{11}}{12}}$$
$$= \frac{48-49}{144} \times \frac{144}{11}$$
$$\rho = \frac{-1}{11}$$

2) Let X & Y have the joint pdf

$$f(x,y) = \begin{cases} e^{-x-y}, & 0 < x < \infty,\ 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

Find (i) M.G.F (ii) correlation coefficient between X & Y (iii) Marginal pdf of X & Y.

Soln:

(i) W.K.T $M(t) = E[e^{tx}]$

$$\therefore M_{X,Y}(t_1, t_2) = E[e^{t_1 X + t_2 Y}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f(x,y)\,dx\,dy$$

$$= \int_0^{\infty} e^{(t_1 - 1)x}\,dx \int_0^{\infty} e^{(t_2-1)y}\,dy$$

$$= \int_0^{\infty}\int_0^{\infty} e^{-(1-t_1)x} e^{-(1-t_2)y}\,dx\,dy$$

$$= \left[\frac{e^{-(1-t_1)x}}{-(1-t_1)}\right]_0^{\infty} \left[\frac{e^{-(1-t_2)y}}{-(1-t_2)}\right]_0^{\infty}$$

$$= \left[0 + \frac{1}{1-t_1}\right]\left[0 + \frac{1}{1-t_2}\right]$$

$$= \frac{1}{(1-t_1)(1-t_2)}$$

$$M(t_1, t_2) = \frac{1}{(t_1-1)(t_2-1)}$$

(ii) W.k.T $\rho = \dfrac{E(XY) - E(X)E(Y)}{\sigma_1 \sigma_2}$

$$E(X) = \left\{ \frac{\partial}{\partial t_1}\left[M(t_1, t_2)\right] \right\}_{t_1 = 0,\, t_2 = 0}$$

$$\frac{\partial}{\partial t_1}\left[M(t_1, t_2)\right] = \frac{-1}{(t_1-1)^2(t_2-1)}$$

At $t_1 = 0$, $t_2 = 0$, $E(X) = \dfrac{-1}{(-1)^2(-1)} = 1$

$$\frac{\partial}{\partial t_2}\left[M(t_1,t_2)\right] = \frac{-1}{(t_1-1)(t_2-1)^2}$$

At $t_1 = 0,\ t_2 = 0$

$$E(Y) = 1$$

$$\frac{\partial^2}{\partial t_1 \partial t_2}\left[M(t_1, t_2)\right] = \frac{1}{(t_1-1)^2(t_2-1)^2}$$

At $t_1 = 0,\ t_2 = 0$

$$E(XY) = 1$$

$$\sigma_1 = E(X^2) - \left[E(X)\right]^2$$

$$E(X^2) = \left\{\frac{\partial^2}{\partial t_1^2}\left[M(t_1, t_2)\right]\right\}_{t_1 = t_2 = 0}$$

$$\frac{\partial^2}{\partial t_1^2}\left[M(t_1, t_2)\right] = \frac{2}{(t_1-1)^3(t_2-1)}$$

$$\therefore E(X^2) = 2$$

$$\sigma_1 = 2 - (1)^2 = 1$$

$$E(Y^2) = \left\{\frac{\partial^2}{\partial t_2^2}\left[M(t_1, t_2)\right]\right\}_{t_1 = 0,\, t_2 = 0}$$

$$\frac{\partial^2}{\partial t_2^2}\left[M(t_1, t_2)\right] = \frac{2}{(t_1-1)(t_2-1)^3}$$

$$E(Y^2) = 2$$

$$\therefore \sigma_2 = 2 - 1 = 1$$

$$\rho = \frac{1 - (1)(1)}{1 \cdot 1} = 0$$

$$\rho = 0$$

iii) Marginal pdf of X is

$$M_1(X) = M(t_1, 0)$$

$$= \frac{1}{(t_1-1)(0-1)}$$

$$= \frac{-1}{(t_1-1)}$$

Marginal pdf of Y is

$$M_2(Y) = M(0, t_2)$$

$$= \frac{1}{(0-1)(t_2-1)}$$

$$= \frac{-1}{t_2 - 1}$$

Independent R.V's :-

2m defn The R.V's x_1 and x_2 are said to be independent if $f(x_1, x_2) = f(x_1) f(x_2)$

R.V's that are not independent are said to be dependent.

Note:
If x and y are independent then $E(X.Y) = E(X).E(Y)$.

problems :

1) S.T the random variables x_1 and x_2 with join pdf $f(x_1, x_2) = \begin{cases} 12 x_1 x_2 (1-x_2), & 0 < x_1 < 1, \; 0 < x_2 < 1 \\ 0 & \text{otherwise} \end{cases}$

are independent

soln: To prove: x_1 and x_2 are independent

ie, To prove:

$$f(x_1, x_2) = f(x_1) \cdot f(x_2)$$

The marginal pdf of x_1 is

$$f(x_1) = \int_{x_2} f(x_1, x_2)\, dx_2$$

$$= \int_0^1 12 x_1 x_2 (1-x_2)\, dx_2$$

$$= 12 x_1 \left[\frac{x_2^2}{2} - \frac{x_3^3}{3}\right]_0^1$$

$$= 12x_1\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 12x_1\left(\frac{1}{6}\right)$$

$$= 2x_1$$

The marginal pdf of X_2 is

$$f(x_2) = \int_{x_1} f(x_1, x_2)\,dx_1$$

$$= \int_0^1 12x_1x_2(1-x_2)\,dx_1$$

$$= 12x_2(1-x_2)\left[\frac{x_1^2}{2}\right]_0^1$$

$$= 6x_2(1-x_2)$$

$$f(x_1)f(x_2) = 2x_1 \cdot 6x_2(1-x_2)$$

$$= 12x_1x_2(1-x_2)$$

$$= f(x_1, x_2)$$

$$\therefore f(x_1, x_2) = f(x_1)f(x_2)$$

Hence X_1 and X_2 are independent.

2) Let the r.v X_1 and X_2 have joint pdf

$$f(x_1, x_2) = \begin{cases} 2e^{-x_1-x_2} & , 0 < x_1 < x_2,\ 0 < x_2 < \infty \\ 0 & , \text{otherwise} \end{cases}$$

S.T X_1 and X_2 are dependent.

Soln:-

$$f(x_1) = \int_{x_2} f(x_1, x_2)\,dx_2$$

$$= \int_0^\infty 2e^{-x_1-x_2}\,dx_2$$

$$= 2e^{-x_1}\left[-e^{-x_2}\right]_0^\infty$$

$$= 2e^{-x_1}$$

$$f(x_2) = \int_{x_1} f(x_1, x_2)\,dx_1$$

$$= \int_0^{x_2} 2e^{-x_1 - x_2}\, dx_1$$

$$= 2e^{-x_2}\left[-e^{-x_1}\right]_0^{x_2}$$

$$= 2e^{-x_2}\left[1 - e^{-x_2}\right]$$

$$= 2e^{-x_2} - 2e^{-2x_2}$$

$$f(x_1) f(x_2) = 2e^{-x_1} \cdot 2(e^{-x_2} - e^{-x_2})$$

$$\neq 2e^{-x_1 - x_2} = f(x_1, x_2)$$

$$\therefore f(x_1, x_2) \neq f(x_1) f(x_2)$$

$\therefore$ The R.V X_1 and X_2 are dependent.

3) Let X_1 and X_2 be independent r.v with marginal pdf $f_1(x_1)$ and $f_2(x_2)$ respectively, then P.T $P(a < X_1 < b,\ c < X_2 < d) = P(a < X_1 < b)\, P(c < X_2 < d)$.

for every a, b, c, d are constants.

Soln:-

case (i) : X_1 and X_2 is a continuous R.V

$$p(a < X_1 < b,\ c < X_2 < d) = \int_c^d \int_a^b f(x_1, x_2)\, dx_1\, dx_2$$

Here X_1 and X_2 are independent random variables

ie., $f(x_1, x_2) = f(x_1) \cdot f(x_2)$

Taking integration on both sides,

$$\int_c^d \int_a^b f(x_1, x_2)\, dx_1\, dx_2 = \int_a^b f(x_1)\, dx_1 \int_c^d f(x_2)\, dx_2$$

$$P(a < X_1 < b,\ c < X_2 < d) = p(a < X_1 < b) \cdot P(c < X_2 < d$$

case (ii) : Discrete type

$$P(a < X_1 < b,\ c < X_2 < d) = \sum_a^b \sum_c^d f(x_1, x_2)$$

Here X_1 and X_2 independent R.V's

$\therefore f(x_1, x_2) = f(x_1) \cdot f(x_2)$

Taking summation on bs,

$$\sum_a^b \sum_c^d f(x_1, x_2) = \sum_a^b f(x_1) \sum_c^d f(x_2).$$

$$p(a < x_1 < b,\ c < x_2 < d) = p(a < x_1 < b) \cdot P(c < x_2 < d).$$

4) Let the pdf of x and y is

$$f(x,y) = \begin{cases} e^{-x-y} & , 0 < x < \infty,\ 0 < y < \infty \\ 0 & , \text{elsewhere} \end{cases}$$

Let $U(x,y) = X$, $V(x,y) = Y$, $W(x,y) = XY$

Show that $E[U(x,y)] \cdot E[V(x,y)] = E[W(x,y)]$

soln:-

$$E[U(x,y)] = E(X)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy$$

$$= \int_0^{\infty} \int_0^{\infty} x \cdot e^{-x-y}\, dx\, dy$$

$$= \int_0^{\infty} x e^{-x} dx \int_0^{\infty} e^{-y} dy$$

$$= \left[-xe^{-x} - e^{-x}\right]_0^{\infty} \left[-e^{-y}\right]_0^{\infty}$$

$$= 1 \cdot 1$$

$$= 1$$

$$E[U(x,y)] = 1$$

III^{ly} $E[V(x,y)] = 1.$

$$E[W(x,y)] = E(XY)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty xy e^{-x-y}\, dx\, dy.$$

$$= \int_0^\infty x e^{-x}\, dx \int_0^\infty y e^{-y}\, dy$$

$$= \left[-xe^{-x} - e^{-x} \right]_0^\infty \left[-ye^{-y} - e^{-y} \right]_0^\infty$$

$$= 1 \cdot 1$$

$$= 1$$

$$E[U(x,y)] \cdot E[V(x,y)] = 1 \cdot 1$$

$$= 1$$

$$= E[W(x,y)]$$

5) Let $f(x_1, x_2) = \begin{cases} \frac{1}{16}, & \text{where } x_1 = 1,2,3,4 \\ & \phantom{\text{where }} x_2 = 1,2,3,4 \\ 0 & \text{elsewhere} \end{cases}$

be the joint pdf of X_1 and X_2. Show that X_1 and X_2 are independent.

Soln:-

Given

x_2 \ x_1	1	2	3	4	$f_{X_2}(x_2)$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$4/16$
2	$1/16$	$1/16$	$1/16$	$1/16$	$4/16$
3	$1/16$	$1/16$	$1/16$	$1/16$	$4/16$
4	$1/16$	$1/16$	$1/16$	$1/16$	$4/16$
$f_{X_1}(x_1)$	$4/16$	$4/16$	$4/16$	$4/16$	

The marginal pdf of X_1 is

x_1	1	2	3	4
$f_1(x_1)$	$1/4$	$1/4$	$1/4$	$1/4$

The marginal pdf of X_2 is

x_2	1	2	3	4
$f_2(x_2)$	$1/4$	$1/4$	$1/4$	$1/4$

$$f(x_1) \cdot f(x_2) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} = f(x_1, x_2)$$

$$f(x_1) \cdot f(x_2) = f(x_1, x_2)$$

Hence X_1 and X_2 are independent.

6) Let X_1 and X_2 denote the random variables that have the joint pdf $f(x_1, x_2)$ and the marginal pdf $f_1(x_1)$ and $f_2(x_2)$ respectively. Furthermore, let $M(t_1, t_2)$ denote the joint mgf of the distribution. Then X_1 and X_2 are independent iff

10m

$$M(t_1, t_2) = M(t_1, 0) \cdot M(0, t_2).$$

Soln:-

- Necessary part:

Assume that X_1 and X_2 are linearly independent.

claim $M(t_1, t_2) = M(t_1, 0) \cdot M(0, t_2)$

$$M(t) = E(e^{tX})$$

$$M(t_1, t_2) = E(e^{t_1 X_1 + t_2 X_2})$$

$$= E(e^{t_1 X_1} \cdot e^{t_2 X_2})$$

Since X_1 and X_2 are linearly independent

$$= E(e^{t_1 X_1}) \cdot E(e^{t_2 X_2}) = E(e^{t_1 X_1 + 0}) \cdot E$$

$$M(t_1,t_2) = M(t_1,0)\cdot M(0,t_2).$$

Sufficient part

Assume that $M(t_1,t_2) = M(t_1,0)\cdot M(0,t_2)$

claim x_1 and x_2 are linearly independent

$$f(x_1,x_2) = f_1(x_1)\cdot f_2(x_2)$$

$$M(t_1,0) = E(e^{t_1 X_1})$$

$$= \int_{-\infty}^{\infty} e^{t_1 x_1} f_1(x_1)\,dx_1$$

$$M(0,t_2) = E(e^{t_2 X_2})$$

$$= \int_{-\infty}^{\infty} e^{t_2 x_2} f_2(x_2)\,dx_2$$

$$M(t_1,0)\cdot M(0,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x_1} f_1(x_1)\,dx_1 \cdot e^{t_2 x_2} f_2(x_2)\,dx_2.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 x_2} f_1(x_1)\cdot f_2(x_2)\,dx_1\,dx_2 \rightarrow ①$$

But $M(t_1,t_2)$ is the mgf of x_1 and x_2

$$M(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x_1 + t_2 x_2} f(x_1,x_2)\,dx_1\,dx_2 \rightarrow ②$$

The uniqueness of mgf implies that the two distribution of probability that are described by $f_1(x_1)\cdot f_2(x_2)$ and $f(x_1,x_2)$ are the same.

$\therefore f(x_1,x_2) = f_1(x_1)\cdot f_2(x_2)$

Hence x_1 and x_2 are independent.

UNIT - III

SOME SPECIAL DISTRIBUTIONS

Unit - III

Some special distributions

The binomial and related distributions

Bernoulli distribution:

The pmf of x can be written as

$$p(x) = p^{x}(1-p)^{1-x}, \quad x = 0, 1$$

and we say that x has a Bernoulli distribution.

Mean and variance of a Bernoulli distribution

W.K.T Bernoulli distribution is

$$p(x) = p^{x}(1-p)^{1-x}, \quad x = 0, 1$$

$$E(X) = \sum_{x=0}^{1} x\, p(x)$$

$$= 0 + p^{1}(1-p)^{0}$$

$$E(X) = p$$

$$E(X^2) = \sum_{x=0}^{1} x^2 p(x)$$

$$= \sum_{x=0}^{1} x^2 p^{x}(1-p)^{1-x}$$

$$= p^{1}(1-p)^{1-1}$$

$$E(X^2) = p$$

$\therefore$ Mean $= E(X) = p$

$$\text{variance} = E(X^2) - [E(X)]^2$$

$$= p - p^2$$

$$= p(1-p)$$

Binomial distribution:

A random variable x that has the pmf

$$P(x) = \begin{cases} nC_x p^{x}(1-p)^{n-x}, & x = 0, 1, 2, \dots, n \\ 0, & \text{elsewhere} \end{cases}$$

is said to have a Binomial distribution and any such $p(x)$ is called a binomial pmf.

A binomial distribution is denoted by $b(n, p)$.

The constants n and p are called the parameters of the binomial distribution.

Derivation of the pdf of a binomial distribution

W.K.T the binomial distribution is

$$f(x) = \begin{cases} nC_x p^x (1-p)^{n-x} & x = 0, 1, 2, \ldots, n \\ 0 & \text{elsewhere} \end{cases}$$

To prove : $\sum_{x=0}^{n} f(x) = 1$

$$\sum_{x=0}^{n} f(x) = \sum_{x=0}^{n} nC_x p^x (1-p)^{n-x}$$

$$= (1-p)^n + nC_1 p^1 (1-p)^{n-1} + nC_2 p^2 (1-p)^{n-2} + \ldots + p^n$$

$$= [1-p+p]^n \quad [\because (a+b)^n = a^n + nC_1 a^{n-1} b + \ldots + b^n]$$

$$= 1$$

M.G.F of a binomial distribution:

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x (1-p)^{n-x}$$

$$= (1-p)^n + nC_1 (pe^t)(1-p)^{n-1} + \ldots + (pe^t)^n$$

$$= (1-p+pe^t)^n$$

$$M(t) = (q+pe^t)^n \quad [\because p+q=1]$$

Mean and variance of a binomial distribution

$$M'(t) = n(q+pe^t)^{n-1}\, pe^t$$
$$= npe^t\,(q+pe^t)^{n-1}$$
$$M''(t) = np[e^t(n-1)(q+pe^t)^{n-2}pe^t + e^t(q+pe^t)^{n-1}]$$
$$E(x) = [M'(t)]_{t=0}$$
$$= np(q+p)^{n-1}$$
$$= np \qquad [\because p+q=1]$$
$$E(x^2) = [M''(t)]_{t=0}$$
$$= np[(n-1)(q+p)^{n-2}p + (q+p)^{n-1}]$$
$$= np[(n-1)p+1] \qquad [\because p+q=1]$$
$$= np[np-p+1]$$
$$= n^2p^2 - np^2 + np$$
$$\text{Mean} = E(x)$$
$$= np$$
$$\text{variance} = E(x^2) - [E(x)]^2$$
$$= n^2p^2 - np^2 + np - n^2p^2$$
$$= np(1-p)$$
$$\therefore \text{mean} = np, \quad \text{variance} = npq$$

pblms

1) Let x be the number of heads in $n=7$ independent tosses of an unbaised coin. Find i) pmf of x ii) MGF iii) mean and variance iv) $P(0 \le x \le 1)$ v) $p(x=5)$

soln:- Given $n=7$

p = probability of getting heads

ie) $p = \frac{1}{2}$

$\therefore q = 1-p = 1-1/2 = 1/2$

Here x is a binomial distribution

$f(x) = n_{C_x} p^x (1-p)^{n-x} \quad x=0,1,2,\dots,n$

i) The pdf of x is

$$f(x) = 7_{C_x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{7-x} \quad x=0,1,2,\dots,7$$

ii) $M_x(t) = (q+pe^t)^7$

$$= \left(\frac{1}{2} + \frac{1}{2}e^t\right)^7$$

iii) Mean $= np = 7 \times \frac{1}{2} = \frac{7}{2}$

variance $= npq = 7 \times \frac{1}{2} \times \frac{1}{2} = \frac{7}{4}$

iv) $P(0 \le X \le 1) = P(X=0) + P(X=1)$

$$= 7C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^7 + 7C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^6$$

$$= \frac{1}{2^7} + \frac{7}{2^7}$$

$$= \frac{1}{2^4}$$

$$= \frac{1}{16}$$

v) $P(X=5) = 7C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2$

$$= \frac{21}{2^7}$$

2) If the mgf of a R.V X is $M(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^5$. Find the pdf of X, mean and variance.

Soln:-

Here $p = \frac{1}{3}$, $q = \frac{2}{3}$, $n=5$

Here x is a binomial distribution.

$$\therefore f(x) = 5C_x\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{5-x}, \quad x=0,1,2,\dots,5$$

$$\text{mean} = np = 5 \times \frac{1}{3} = 5/3$$

$$\text{Variance} = npq = 5 \cdot 1/3 \cdot 2/3 = 10/9$$

3) If the MGF of a R.V is $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$, then show that $P(\mu-2\sigma < X < \mu+2\sigma) = \sum_{x=1}^{5} 9c_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$

Soln:-

$n = 9, \; p = 1/3, \; q = \frac{2}{3}$

Here X is a binomial distribution

$$f(x) = nc_x \, p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n$$

$$\text{mean} = np = 9 \cdot 1/3 = 3$$

$$\text{variance} = npq = 9 \cdot 1/3 \cdot 2/3$$

$$\sigma^2 = 2$$

$$\sigma = \sqrt{2}$$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(3 - 2\sqrt{2} < X < 3 + 2\sqrt{2})$$

$$= P(0.172 < X < 5.828)$$

$$= P(X = 1, 2, \ldots, 5)$$

$$= \sum_{x=1}^{5} 9c_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$$

4) If the mgf of a R.V is given by $\left(\frac{1}{3}+\frac{2}{3}e^t\right)^5$. Find $P[X=2 \text{ or } X=3]$.

proof:-

$n = 5, \; p = 2/3, \; q = 1/3$

Here X is a binomial distribution.

$$f(x) = nc_x \, (p)^n (1-p)^{n-x} \quad x = 0, 1, \ldots, n$$

$$P[X=2 \text{ or } X=3] = P(X=2) + P(X=3)$$

$$= 5c_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^3 + 5c_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2$$

$$= 10 \times \frac{4}{9} \times \frac{1}{27} + 10 \times \frac{8}{27} \times \frac{1}{9}$$

$$= \frac{120}{27\times 9}$$

$$= \frac{40}{81}$$

5) If x is a binomial distribution then S.T (i) $E\left(\frac{x}{n}\right) = p$ (ii) $E\left(\frac{x}{n} - p\right)^2 = \frac{p(1-p)}{n}$.

Soln:-

i) $E\left(\frac{x}{n}\right) = \frac{E(x)}{n} = \frac{np}{n} = p$

ii) $E\left(\frac{x}{n} - p\right)^2 = E\left[\frac{x^2}{n^2} + p^2 - 2\frac{xp}{n}\right]$

$$= \frac{1}{n^2}E(x^2) + p^2 - \frac{2p}{n}E(x)$$

$$= \frac{1}{n^2}E(x^2) + p^2 - 2\frac{p}{n}np$$

$$= \frac{1}{n^2}E(x^2) + p^2 - 2p^2$$

$$= \frac{1}{n^2}E(x^2) - p^2$$

W.K.T $\sigma^2 = E(x^2) - [E(x)]^2 \Rightarrow E(x^2) = \sigma^2 + n^2p^2$

$\therefore E\left(\frac{x}{n} - p\right)^2 = \frac{1}{n^2}(\sigma^2 + n^2p^2) - p^2$

$$= \frac{\sigma^2}{n^2} + \frac{n^2p^2}{n^2} - p^2$$

$$= \frac{npq}{n^2} + p^2 - p^2$$

$$= \frac{\cancel{n}pq}{n^{\cancel{2}}}$$

$E\left(\frac{x}{n} - p\right)^2 = \frac{p(1-p)}{n}$ Binomial dist is denoted by $B(n,p)$

6) Let x be the Bd of $B(2,p)$ and y be the B.d of $B(4,p)$. If $P(X \geq 1)$ is $\frac{5}{9}$ then find $P(Y \geq 1)$.

Soln:

$f(x) = nc_x\, p^x q^{n-x}$, $x = 0,1,2,\ldots$

Given $P(X \geq 1) = \frac{5}{9} \Rightarrow 1 - P(X<1) = \frac{5}{9}$

$$\Rightarrow 1 - P(X=0) = \frac{5}{9}$$

$$\Rightarrow P(X=0) = 1 - 5/9$$

$$P(X=0) = \frac{4}{9}$$

$$q^2 = \frac{4}{9} \Rightarrow q = 2/3$$

$$\therefore P = 1/3$$

$$\therefore n=4, \; P = 1/3, \; q = 2/3$$

$$P(Y \geq 1) = 1 - P(Y < 1)$$

$$= 1 - P(Y=0)$$

$$= 1 - (2/3)^4$$

$$= 1 - 16/81$$

$$= \frac{81-16}{81}$$

$$= \frac{65}{81}$$

7) Let Y be the no of n independent repetitions of a random experiment having the probability of success $P = 1/4$. Determine the smallest value of n so that $P(1 \leq Y) \geq 0.70$.

Soln:-

Let $g(y)$ be the pdf of Y.

$$g(y) = nC_y \, p^y q^{n-y}, \; y = 0, 1, \ldots, n$$

$$P = 1/4, \; q = 3/4$$

$$P(1 \leq Y) = 0.7$$

$$1 - P(Y < 1) = 0.7$$

$1 - P(Y=0) = 0.7$

$1 - nC_0\, p^0 q^n = 0.7 \Rightarrow \left(\frac{3}{4}\right)^n = 1 - 0.7$

$\left(\frac{3}{4}\right)^n = 0.3$

$n \log(3/4) = \log 0.3$

$n = 4.1866$

$\therefore n = 4$

8) Let x have a binomial distribution with parameters n, $p = 1/3$ determine the smallest integer n s.t. $P(X \geq 1) \geq 0.85$

soln:-

$1 - P(X<1) \geq 0.85$

$1 - P(X=0) \geq 0.85$

$1 - \left(\frac{2}{3}\right)^n \geq 0.85$

$\left(\frac{2}{3}\right)^n \leq 0.15$

$n \log \frac{2}{3} \leq \log 0.15$

$n \leq 4.6812$ $\quad 4.6786$

$\therefore n = 5$

9) Let the independent r.v.s X_1, X_2 and X_3 have the same pdf $f(x) = \begin{cases} 3x^2, & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

Find the probability that exactly 2 of these 3 variables exceeds $\frac{1}{2}$.

soln:- Givn $f(x) = \begin{cases} 3x^2, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$

P(exactly 2 of these 3 variables exceeds $\frac{1}{2}$)

$= P(X_1 > \frac{1}{2}, X_2 > \frac{1}{2}, X_3 \leq \frac{1}{2}) +$
$P(X_1 > 1/2, X_2 \leq 1/2, X_3 > 1/2) +$
$P(X_1 \leq \frac{1}{2}, X_2 > 1/2, X_3 > 1/2)$

$$P(X > 1/2) = \int_{1/2}^{1} f(x)\,dx$$

$$= [x^3]_{1/2}^{1}$$

$$= 1 - 1/8$$

$$= 7/8$$

$$P(X \le 1/2) = \int_{0}^{1/2} 3x^2\,dx$$

$$= \left[\frac{3x^3}{3}\right]_0^{1/2}$$

$$= 1/8$$

$\therefore$ P(exactly 2 of these 3 variables exceed $\frac{1}{2}$)

$$= \frac{7}{8} \times \frac{7}{8} \cdot \frac{1}{8} + \frac{7}{8} \cdot \frac{1}{8} \cdot \frac{7}{8} + \frac{1}{8} \cdot \frac{7}{8} \cdot \frac{7}{8}$$

$$= 3 \cdot \frac{49}{512}$$

$$= \frac{147}{512}.$$

10) Let Y be the no. of success throughout any independent repeatable of a random experiment having the probability of success $p = 2/3$. If $n = 3$ compute $P(2 \le Y)$. If $n = 5$ compute $P(3 \not\le Y)$.

Soln:-

$$p = \frac{2}{3}, \quad q = \frac{1}{3}$$

i) $n = 3$

$$P(2 \le Y) = 1 - P(Y < 2)$$

$$= 1 - [P(Y=0) + P(Y=1)]$$

$$= 1 - \left[\left(\frac{1}{3}\right)^3 + 3 \cdot \frac{2}{3}\left(\frac{1}{3}\right)^2\right]$$

$$= 1 - \left[\frac{1}{3^3} + \frac{2}{3^2}\right]$$

$$= 1 - \left[\frac{1+6}{3^3}\right]$$

$$= 1 - \frac{7}{3^3}$$

$$= \frac{27-7}{27}$$

$$= \frac{20}{27}$$

ii) $n=5$

$$P(3 \le Y) = 1 - [P(Y=0) + P(Y=1) + P(Y=2)]$$

$$= 1 - \left[\left(\frac{1}{3}\right)^5 + 5 \cdot \frac{2}{3} \left(\frac{1}{3}\right)^4 + 10\left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^3 \right]$$

$$= 1 - \left[\frac{1}{243} + \frac{10}{243} + \frac{40}{243}\right]$$

$$= 1 - \frac{51}{243}$$

$$= \frac{243-51}{243} = \frac{192}{243}$$

$$= \frac{64}{81}.$$

11) If $x = r$ is a unique mode of a distribution i.e., $b(n,p)$ s.t $(n+1)p - 1 < r < (n+1)p$

[Hint : Determine the values of x for which the ratio $\frac{f(x+1)}{f(x)} > 1$]

5m

soln:-

Givn $x = r$ is a unique mode of a distribution.

$$f(x-1) < f(x) < f(x+1)$$

Take $f(x-1) < f(x)$

$$\frac{f(x-1)}{f(x)} < 1$$

$$\frac{f(r-1)}{f(r)} < 1$$

$$\frac{nC_{r-1}\, p^{r-1} q^{n-r+1}}{nC_r\, p^r q^{n-r}} < 1$$

$$\frac{\frac{n!}{(r-1)!(n-r+1)!} p^{-1} q}{\frac{n!}{r!(n-r)!}} < 1$$

$$\frac{rq}{(n-r+1)p} < 1$$

$$r(1-p) < p(n-r+1)$$

$$r - rp < np - pr + p$$

$$r < p(n+1) \rightarrow (1)$$

$$r-1 < p(n+1)-1$$

$$\therefore p(n+1)-1 < r \rightarrow (2) \quad [\because r-K r]$$

From (1) and (2),

$$p(n+1)-1 < r < p(n+1)$$

Law of large numbers

If Y is $b(n,p)$ and the ratio $\frac{Y}{n}$ is called the relative frequency of success. Then for every $\varepsilon > 0$, $\lim_{n\to\infty} P\left[\left|\frac{Y}{n} - p\right| \geq \varepsilon\right] = 0$ and $\lim_{n\to\infty} P\left[\left|\frac{Y}{n} - p\right| < \varepsilon\right] = 1$.

proof:- Let the r.v Y be equal to the no of success throughout n independent repetitions of a random experiment with probability p of success i.e., Y is $b(n,p)$.

$$P\left[\left|\frac{Y}{n} - p\right| \geq \varepsilon\right] = P\left[|Y - np| \geq n\varepsilon\right]$$

$$= P\left[|Y-\mu| \geq n\varepsilon \frac{\sigma}{\sigma}\right]$$

$$= P\left[|Y-\mu| \geq \frac{n\varepsilon}{\sqrt{npq}} \cdot \sigma\right]$$

$$= P\left[|Y-\mu| \geq \frac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}}\,\sigma\right] \rightarrow (1)$$

W.K.T chebyshev's inequality is

$$P\left[|X-\mu| \geq k\sigma\right] \leq \frac{1}{k^2}$$

Here $k = \dfrac{\sqrt{n}\,\varepsilon}{\sqrt{p(1-p)}}$

(1) becomes

$$P\left[\left|\frac{Y}{n}-p\right| \geq \varepsilon\right] \leq \frac{1}{\frac{n\varepsilon^2}{p(1-p)}}$$

$$= \frac{p(1-p)}{n\varepsilon^2}$$

$$\lim_{n\to\infty} P\left[\left|\frac{Y}{n}-p\right| \geq \varepsilon\right] \leq \lim_{n\to\infty} \frac{p(1-p)}{n\varepsilon^2}$$

$$= 0$$

$$\lim_{n\to\infty} P\left[\left|\frac{Y}{n}-p\right| \geq \varepsilon\right] = 0$$

$$P\left[\left|\frac{Y}{n}-p\right| < \varepsilon\right] = 1 - P\left[\left|\frac{Y}{n}-p\right| \geq \varepsilon\right]$$

$$\lim_{n\to\infty} P\left[\left|\frac{Y}{n}-p\right| < \varepsilon\right] = 1 - \lim_{n\to\infty} P\left[\left|\frac{Y}{n}-p\right| \geq \varepsilon\right]$$

$$= 1 - 0$$

$$= 1.$$

Negative binomial distribution

A distribution with a pmf of the form

$$g(y) \text{ (or) } P_Y(y) = \begin{cases} (y+r-1)C_{r-1}\, p^r (1-p)^y, & y = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

is called a negative binomial distribution and any such pmf $P_Y(y)$ is called a negative binomial pmf.

MGF of a negative binomial distribution

The negative binomial distribution is

$$g(y) = (y+r-1)C_{r-1}\, p^r (1-p)^y, \quad y = 0, 1, \ldots$$

$$M(t) = E(e^{tY})$$

$$= \sum_{y=0}^{\infty} e^{ty} g(y)$$

$$= \sum_{y=0}^{\infty} e^{ty} \left[(y+r-1)C_{r-1}\, p^r (1-p)^y \right]$$

$$= p^r \sum_{y=0}^{\infty} (y+r-1)C_{r-1} \left[e^t (1-p) \right]^y$$

$$= p^r \left[1 + rC_{r-1} \left[e^t (1-p) \right] + (r+1)C_{r-1} \left[e^t (1-p) \right]^2 + \ldots \right]$$

$$M(t) = p^r \left[1 - e^t (1-p) \right]^{-r}.$$

Mean and variance of a negative binomial distribution

$$M'(t) = p^r (-r) \left[1 - e^t (1-p) \right]^{-r-1} (-e^t (1-p))$$

$$= p^r \left[r(1-p) e^t \left[1 - e^t (1-p) \right]^{-r-1} \right]$$

$$M''(t) = p^r r (1-p) e^t \left[1 - e^t (1-p) \right]^{-r-1} + p^r r (1-p) e^t (-r-1) \left[1 - e^t (1-p) \right]^{-r-2} (-e^t (1-p))$$

$$= p^r r (1-p) e^t \left[1 - e^t (1-p) \right]^{-r-1} + p^r r (1-p)^2 (r+1) (e^t)^2 \left[1 - e^t (1-p) \right]^{-r-2}$$

At $t=0$,

$$M'(t) = p^r r(1-p)\, p^{-r-1}$$

$$E(X) = \frac{r(1-p)}{p}$$

$$M''(t) = p^r r(1-p)p^{-r-1} + p^r r(1-p)^2 (r+1) p^{-r-2}$$

$$E(X^2) = \frac{r(1-p)}{p} + \frac{r(r+1)(1-p)^2}{p^2}$$

$$\text{Mean} = \frac{r(1-p)}{p}$$

$$\text{var} = E(X^2) - [E(X)]^2$$

$$= \frac{r(1-p)}{p} + \frac{r(r+1)(1-p)^2}{p^2} - \frac{r^2(1-p)^2}{p^2}$$

$$= \frac{r(1-p)}{p} + \frac{r(1-p)^2}{p^2}[r+1-r]$$

$$= \frac{r(1-p)}{p}\left[1 + \frac{1-p}{p}\right]$$

$$= \frac{r(1-p)}{p^2}$$

Geometric distribution:

The distribution with pdf of th form

$$g(y) = \begin{cases} p(1-p)^y, & y = 0,1,2,\dots \\ 0 & \text{elsewhere} \end{cases}$$ is called

the geometric distribution.

M.G.F of a geometric distribution

The geoemetric distribution is

$$g(y) = p(1-p)^y \quad , y = 0,1,2,\dots$$

$$M(t) = E(e^{tY})$$

$$= \sum_{y=0}^{\infty} e^{ty}\, g(y)$$

$$= \sum_{y=0}^{\infty} e^{ty}\, p(1-p)^{y}$$

$$= p \sum_{y=0}^{\infty} \left[e^{t}(1-p)\right]^{y}$$

$$= p\left[1 + e^{t}(1-p) + \left[e^{t}(1-p)\right]^{2} + \left[e^{t}(1-p)\right]^{3} + \cdots\right]$$

$$\cancel{= p\left[1 - e^{t}(1-p)\right]^{-1}}$$

$$M(t) = p\left[1 - e^{t}(1-p)\right]^{-1}$$

Mean and variance of a geometric distribut

$$M'(t) = -p\left[1 - e^{t}(1-p)\right]^{-2}\left(-e^{t}(1-p)\right)$$

$$= p e^{t}(1-p)\left[1 - e^{t}(1-p)\right]^{-2}$$

$$M''(t) = p e^{t}(1-p)(-2)\left[1 - e^{t}(1-p)\right]^{-3}\left(-e^{t}(1-p)\right) + p e^{t}(1-p)\left[1 - e^{t}(1-p)\right]^{-2}$$

$$= 2p e^{2t}(1-p)^{2}\left[1 - e^{t}(1-p)\right]^{-3} + p e^{t}(1-p)\left[1 - e^{t}(1-p)\right]^{-2}$$

At $t = 0$,

$$M'(t) = p(1-p)\left[1 - 1 + p\right]^{-2}$$

$$= p(1-p)\,p^{-2}$$

$$= \frac{p(1-p)}{p^{2}}$$

$$E(X) = \frac{1-p}{p}$$

$$M''(t) = 2p(1-p)^{2}\left[1 - 1 + p\right]^{-3} + p(1-p)\left[1 - 1 + p\right]^{-2}$$

$$= 2p(1-p)^2 p^{-3} + p(1-p)\, p^{-2}$$

$$E(x^2) = \frac{2(1-p)^2}{p^2} + \frac{(1-p)}{p}$$

$$\text{var} = E(x^2) - [E(x)]^2$$

$$= \frac{2(1-p)^2}{p^2} + \frac{(1-p)}{p} - \left(\frac{1-p}{p}\right)^2$$

$$= \frac{2(1-p)^2}{p^2} + \frac{(1-p)}{p} - \frac{(1-p)^2}{p^2}$$

$$= \frac{(1-p)^2}{p^2} + \frac{(1-p)}{p}$$

$$= \frac{(1-p)}{p^2}(1-p+p)$$

$$\text{var} = \frac{1-p}{p^2}$$

Note:

put $r=1$ in the negative binomial distribution, we get the geometric distribution

$$M.G.F = p[1-e^t(1-p)]^{-1}$$

$$\text{Mean} = \frac{1-p}{p}$$

$$\text{variance} = \frac{1-p}{p^2}$$

Poisson distribution

A R.V that has the pdf

$$f(x) = \begin{cases} \dfrac{e^{-m} m^x}{x!}, & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$ is said to

have a poisson distribution with parameter m and any such $f(x)$ is called a poisson pdf

MGF : Mean, var :

$$M(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-m} m^x}{x!} = e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!}$$

$$= e^{-m}\left[1 + \frac{me^t}{1!} + \frac{(me^t)^2}{2!} + \cdots\right]$$

$$= e^{-m}\left[e^{me^t}\right] \quad \left[\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right]$$

$$= e^{-m+me^t} = e^{m(e^t-1)}$$

$$M'(t) = e^{m(e^t-1)}(me^t)$$

$$M''(t) = e^{m(e^t-1)}(me^t) + e^{m(e^t-1)}(me^t)^2$$

$$\mu = M'(0) = m$$

$$M''(0) = m + m^2$$

$$\sigma^2 = m + m^2 - m^2 = m$$

$\therefore$ Mean = variance = m

pblms

1) Let x have a poisson distribution with $\mu = 2$. Find $P(X \geq 1)$ and var

soln:-

W.K.T the pdf of poisson distribution is

$$f(x) = \frac{e^{-m} m^x}{x!}, \quad x = 0, 1, \ldots$$

Gn mean $= \mu = 2 = m$

$$f(x) = \frac{e^{-2} 2^x}{x!}, \quad x = 0, 1, \ldots$$

$$\text{var} = m = 2$$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - \frac{e^{-2} \cdot 2^0}{0!}$$

$$= 0.865$$

2) If the mgf of a R.V x is $e^{4(e^t-1)}$ find $p(x=3)$

soln:- Givn $m=4$

$$f(x) = \frac{e^{-4} 4^x}{x!}, \quad x = 0, 1, \dots$$

$$p(x=3) = \frac{e^{-4} \cdot 4^3}{3!} = \frac{32}{3} e^{-4} = 0.1954$$

3) If the R.V x has a p. dist $\ni$: $p(x=1) = p(x=2)$. Find $p(x=4)$.

soln:- W.K.T the pdf of a p. dist is

$$f(x) = \frac{e^{-m} \cdot m^x}{x!}, \quad x = 0, 1, \dots$$

Givn $p(x=1) = p(x=2)$

ie., $$e^{-m} \cdot m = \frac{e^{-m} m^2}{2!} \Rightarrow m = 2$$

$$p(x=4) = \frac{e^{-2} \cdot 2^4}{4!} = \frac{2e^{-2}}{3} = 0.0902$$

4) The Mgf of a R.V x is $e^{4(e^t-1)}$. S.T $p(\mu - 2\sigma < x < \mu + 2\sigma) = 0.931$

soln:- W.K.T, M(t) of poisson distribution is

$$M(t) = e^{m(e^t-1)}$$

Givn $$M(t) = e^{4(e^t-1)} \Rightarrow m = 4.$$

$$\mu = 4$$

$$\sigma^2 = 4.$$

$$\sigma = 2.$$

$$p(4 - 2(2) < x < 4 + 2(2))$$

$$= p(0 < x < 8)$$

$$= \sum_{x=1}^{7} \frac{e^{-m} m^x}{x!}$$

$$= e^{-4}\left[\frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \frac{4^6}{6!} + \frac{4^7}{7!}\right]$$

$$= e^{-4}\left[\frac{4}{1} + 8 + \frac{32}{3} + \frac{32}{3} + \frac{128}{15} + \frac{256}{45} + \frac{1024}{315}\right]$$

$$= e^{-4}\left[\frac{1260 + 2520 + 3360 + 3360 + 2688 + 1792 + 1024}{315}\right]$$

$$= e^{-4}\left[\frac{16004}{315}\right]$$

$$= (0.0183)(50.8063)$$

$$= 0.9298$$

$$= 0.931$$

Gamma distribution:

A R.v that has the pdf of the form

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, & 0 < x < \infty,\ \alpha > 0, \beta > 0, \Gamma(\alpha) > 0 \\ 0 & \text{elsewhere} \end{cases}$$

is said to have a gamma distribution with parameters α and β.

Note:

1) Gamma function : $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx$
2) $\Gamma(1) = 1$
3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
4) $\Gamma(n) = (n-1)\Gamma(n-1)$
5) $\Gamma(n) = (n-1)!$

Mgf, mean, var:

$$M(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}\,dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x(1-\beta t)/\beta} dx,$$

put $y = \frac{x(1-\beta t)}{\beta} \Rightarrow x = \frac{\beta y}{1-\beta t} \Rightarrow dx = \frac{\beta}{1-\beta t} dy$

x	0	∞
y	0	∞

y varies from 0 to ∞

$$M(t) = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} \left(\frac{\beta y}{1-\beta t}\right)^{\alpha-1} e^{-y} \frac{\beta}{1-\beta t} dy$$

$$= \left(\frac{1}{1-\beta t}\right)^\alpha \cdot \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y} dy$$

$$= \frac{1}{(1-\beta t)^\alpha} = (1-\beta t)^{-\alpha}, \quad t < \frac{1}{\beta}$$

$$M'(t) = (-\alpha)(1-\beta t)^{-\alpha-1}(-\beta)$$

$$= \alpha\beta(1-\beta t)^{-\alpha-1}$$

$$M''(t) = \alpha\beta(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)$$

$$= -\alpha\beta^2(-\alpha-1)(1-\beta t)^{-\alpha-2}$$

$$E(X) = M'(0) = \alpha\beta$$

$$E(X^2) = M''(0) = \alpha(\alpha+1)\beta^2$$

$$\mu = \alpha\beta$$

$$\text{var} = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2$$

$$= \alpha\beta^2$$

Pblm:1

If $(1-2t)^{-6}$, $t < \frac{1}{2}$ is the mgf of the R.V x. find mean, var.

soln:

Here $\alpha = 6$, $\beta = 2$.

$$\therefore f(x) = \frac{1}{\Gamma(6) 2^6} x^5 e^{-x/2}$$

$$\text{mean} = \alpha\beta = (6)(2) = 12$$

$$\text{var} = \alpha\beta^2 = (6)(4) = 24$$

Chisquare distribution:

consider the gamma distribution in which $\alpha = r/2$, where r is a positive integer and $\beta = 2$. A R.v X of the continuous type that has the pdf

$$f(x) = \begin{cases} \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, & 0<x<\infty \\ 0 & \text{elsewhere} \end{cases}$$

is said to have a chi-square distribution and any $f(x)$ is called a chi-square pdf

Mgf : $M(t) = (1-2t)^{-r/2}$, $t < 1/2$

Mean : $r/2 \times 2 = r$

Var : $r/2 \times 4 = 2r$

ie., X is the R.v has a chi-square distribution with r degrees of freedom.

pblms

1) If X has the pdf $f(x) = \begin{cases} \frac{1}{4} x e^{-x/2}, & 0<x<\infty \\ 0 & \text{elsewhere} \end{cases}$

Find mgf, mean, var.

soln:- Given $f(x) = \begin{cases} \frac{1}{4} x e^{-x/2}, & 0<x<\infty \\ 0 & \text{elsewhere} \end{cases}$

Here $\beta = 2$, $\alpha = 2$ $\qquad f(x) = \begin{cases} \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, & 0<x<\infty \\ 0 & \text{else where} \end{cases}$

$\therefore X$ is a chisquare distribution

$\alpha = r/2 \Rightarrow r = 4$

$\therefore X$ is $\chi^2(4)$

MGF : $M(t) = (1-\beta t)^{-\alpha} = (1-2t)^{-2}$

mean $= r = 4$, var $\sigma^2 = 2r = 8$.

2) Let X be $\chi^2(10)$. Then find $P(3.25 \le X \le 20.5)$

soln:- Given $r=10$

$$P(3.25 \le X \le 20.5) = P(X \le 20.5) - P(X \le 3.25)$$

$$= 0.975 - 0.025 \quad \text{[From table]}$$

$$= 0.95$$

3) Let X have a gamma distribution with $\alpha = r/2$, where r is a positive integer and $\beta > 0$. Define the R.V $Y = \frac{2X}{\beta}$. Find the pdf of Y.

5m

soln:-

W.K.T the cdf of Y is $G(y) = P(Y \le y)$

$$= P\left(\frac{2X}{\beta} \le y\right)$$

$$= P\left(X \le \frac{\beta y}{2}\right)$$

$$G(y) = \int_0^{\beta y/2} \frac{1}{\Gamma(r/2)\beta^{r/2}} x^{r/2 - 1} e^{-x/\beta}\, dx$$

$$= \int_0^{x} \frac{1}{\Gamma(r/2)\beta^{r/2}} \left(\frac{\beta y}{2}\right)^{r/2 - 1} e^{-\beta y/2\beta} \frac{\beta\, dy}{2}$$

$$\left[\text{put } x = \frac{\beta y}{2}\right]$$

$$= \int_0^{x} \frac{1}{\Gamma(\frac{r}{2})\, 2^{r/2}} \cdot y^{r/2 - 1} e^{-y/2}\, dy$$

The pdf of Y is

$$g(y) = G'(y) = \frac{1}{\Gamma(r/2)\, 2^{r/2}}\, y^{r/2 - 1} e^{-y/2}$$

ie., Y is $\chi^2(r)$.

Normal distribution

We say a R.V X has a normal distribution if its pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad \text{for } -\infty < x < \infty$$

It is written as a $N(\mu, \sigma^2)$ distribution

The R.V z with pdf $f(x) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ has a $N(0,1)$ distribution. z is called a standard normal R.V

Note:

x has a $N(\mu, \sigma^2)$ distribution iff

$z = \frac{x-\mu}{\sigma}$ has a $N(0,1)$ distribution.

standard normal distribution.

MGF, mean, var:

15m (or) 10m

$$M(t) = E[e^{tx}]$$

W.k.T $z = \frac{x-\mu}{\sigma} \Rightarrow x = \sigma z + \mu$

$$\therefore M(t) = E[e^{t(\sigma z + \mu)}]$$

$$= e^{\mu t} E[e^{t\sigma z}]$$

$$E[e^{t\sigma z}] = \int_{-\infty}^{\infty} e^{t\sigma z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2 + t\sigma z} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2t\sigma z + t^2\sigma^2 - t^2\sigma^2)} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-1/2(z - t\sigma)^2} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-1/2 u^2} du$$

[$\because$ put $u = z - \sigma t$
$du = dz$]

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{t^2\sigma^2}{2}} 2 \int_{0}^{\infty} e^{-u^2/2} du$$

[$\because \int$ is an even function]

$$= \frac{2}{\sqrt{2\pi}} e^{t^2\sigma^2/2} \int_{0}^{\infty} w^{-1/2} e^{-w} \frac{dw}{\sqrt{2}}$$

$\Big[$ put $\quad u^2/2 = \omega$

$\quad\quad \cancel{2}u\,du/\cancel{2} = d\omega$

$u^2 = 2\omega \quad\quad du = \dfrac{d\omega}{\sqrt{2}\,\sqrt{\omega}} \Big]$

$u = \sqrt{2}\sqrt{\omega}$

$$= \frac{2}{\sqrt{2}\,\sqrt{2\pi}}\, e^{t^2\sigma^2/2} \int_0^\infty \omega^{1/2-1} e^{-\omega}\, d\omega$$

$$= \frac{1}{\sqrt{\pi}}\, e^{t^2\sigma^2/2}\, \Gamma\left(\tfrac{1}{2}\right)$$

$$= e^{t^2\sigma^2/2} \quad \left[\because \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right]$$

$$\therefore M(t) = e^{\mu t} \cdot e^{t^2\sigma^2/2}$$

$$= e^{\mu t + \sigma^2 t^2/2}$$

$$M'(t) = e^{\mu t + \sigma^2 t^2/2}\,(\mu + \sigma^2 t)$$

$$M''(t) = e^{\mu t + \sigma^2 t^2/2}\,(\sigma^2) + e^{\mu t + \sigma^2 t^2/2}\,(\mu + \sigma^2 t)^2$$

$E(x) = M'(0) = \mu, \quad E(x^2) = \sigma^2 + \mu^2$

Mean $= E(x) = \mu$, var $= E(x^2) - [E(x)]^2$

$= \sigma^2 + \mu^2 - \mu^2$

$= \sigma^2$

pblms

1) If e^{2t+32t^2} is the mgf of x then find mean and var.

soln:-

Givn $M(t) = e^{2t+32t^2}$

W.k.T $\quad M(t) = e^{\mu t + \sigma^2 t^2/2}$

$\mu = 2, \quad \sigma^2 = 64.$

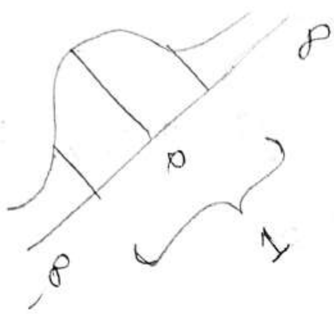
2) Let x be $N(2, 25)$ find $P(0 < x < \infty)$, $P(-8 < x < 1)$

Soln:-

$\mu = 2, \quad \sigma^2 = 25 \Rightarrow \sigma = 5$

$$P(0 < x < 10) = P\left(\frac{0-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{10-\mu}{\sigma}\right)$$

$$= P\left(\frac{-2}{5} < z < \frac{10-2}{5}\right)$$



$$= P(-0.4 < z < 1.6)$$

$$= P(z < 1.6) - P(z < -0.4)$$

$$= \phi(1.6) - \phi(-0.4)$$

$$= \phi(1.6) - [1 - \phi(0.4)]$$

$$= 0.945 - 1 + 0.655$$

$$= 0.6$$

$$P(-8 < x < 1) = P\left(\frac{-8-2}{5} < z < \frac{1-2}{5}\right)$$

$$= P(-2 < z < -0.2)$$

$$= \phi(-0.2) - \phi(-2)$$

$$= 1 - \phi(0.2) - [1 - \phi(2)]$$

$$= 1 - 0.579 - (1 - 0.977)$$

$$= 0.398$$

3) Let x be $N(\mu, \sigma^2)$. Find $P(\mu - 2\sigma < x < \mu + 2\sigma)$

Soln:-

$$P(\mu - 2\sigma < x < \mu + 2\sigma) = P(-2 < z < 2)$$

$$= \phi(2) - \phi(-2)$$

$$= \phi(2) - [1 - \phi(2)]$$

$$= 0.977 - 1 + 0.977$$

$$= 0.954$$

4) If e^{3t+8t^2} is the mgf of x find $p(-1<x<9)$

Home work

soln:-

Given $M(t) = e^{3t+8t^2}$, W.K.T $M(t) = e^{\mu t + \sigma^2 t^2/2}$

Here $\mu = 3$, $\sigma^2 = 16 \Rightarrow \sigma = 4$

$$p(-1<x<9) = P\left(\frac{-1-3}{4} < z < \frac{9-3}{4}\right)$$

$$= p(-1<z<1.5)$$

$$= p(z<1.5) - p(z<-1)$$

$$= \phi(1.5) - \phi(-1)$$

$$= \phi(1.5) - [1-\phi(1)]$$

$$= \phi(1.5) - 1 + \phi(1)$$

$$= 0.9332 - 1 + 0.8413$$

$$= 0.7745$$

4) Let the R.V x have the pdf $p(x) = \begin{cases} \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, & 0<x<\infty \\ 0, & \text{otherwise} \end{cases}$

Find the mean and var of x. $E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$

soln:-

$$E(x) = \int_0^\infty x \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx$$

put $\frac{x^2}{2} = u \Rightarrow \frac{2x\,dx}{2} = du$

when $x=0$, $u=0$

when $x=\infty$, $u=\infty$

$$E(x) = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-u} du = \frac{2}{\sqrt{2\pi}} \left(-e^{-u}\right)_0^\infty = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$$

$$E(x^2) = \int_0^\infty x^2 \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx$$

put $\frac{x^2}{2} = u$

$x = \sqrt{2u}$

$x\,dx = du$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \sqrt{2u}\, e^{-u} dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty u^{3/2-1} e^{-u} du$$

$$= \frac{2}{\sqrt{\pi}} \Gamma(3/2) = \frac{2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$[\because \Gamma(n+1) = n\Gamma(n)]$

$E(x^2) = 1$ $\quad [\because \Gamma(\frac{1}{2}) = \sqrt{\pi}]$

$$Var = 1 - \frac{2}{\pi}$$

$$= \frac{\pi - 2}{\pi}.$$

5) Let X be $N(\mu, \sigma^2)$ so that $P(X<89)=0.90$ and $P(X<94) = 0.95$. Find μ and σ^2.

Soln:-

$$P(X<89) = 0.90 \Rightarrow P\left(z < \frac{89-\mu}{\sigma}\right) = 0.9$$

$$\phi\left(\frac{89-\mu}{\sigma}\right) = 0.90$$

$$\frac{89-\mu}{\sigma} = 1.290 \text{ from table}$$

$$\mu + 1.29\sigma = 89 \rightarrow (1)$$

$$P(X<94) = 0.95 \Rightarrow \mu + 1.65\sigma = 94 \rightarrow (2)$$

From (1) and (2), $\mu = 70.9$, $\sigma = 13.88$

ie) $\mu = 71$, $\sigma = 14$.

Bivariate normal distribution

Defn:-

Let X and Y be the two R.V's consider the function

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-q/2}, \quad -\infty < x < \infty, -\infty < y < \infty$$

rho

where $q = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]$

$\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 \le \rho \le 1$

Derivation of $f(x,y)$ is a pdf.

clearly $f(x,y) \geq 0$

we've to prove that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

W.K.T $f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-q/2}, \quad -\infty < x, y < \infty.$

where $q = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right]$

Now $q(1-\rho^2) = \left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 + \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2$

$= \left[\left(\frac{y-\mu_2}{\sigma_2}\right) - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2$

$q = \frac{1}{1-\rho^2}\left[\left(\frac{y-\mu_2}{\sigma_2}\right) - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2$

$= \frac{1}{1-\rho^2}\left(\frac{y-b}{\sigma_2}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2$

where $b = \mu_2 + \frac{\rho\sigma_2}{\sigma_1}(x-\mu_1)$

Now $f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$

$f_1(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-q/2}\,dy.$

$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp -\left[\frac{\frac{1}{1-\rho^2}\left(\frac{y-b}{\sigma_2}\right)^2 + \left(\frac{x-\mu_1}{\sigma_1}\right)^2}{2}\right] dy$

$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right] \int_{-\infty}^{\infty} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{y-b}{\sigma_2}\right)^2\right] dy$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right] \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}}$$

$$\exp\left[\frac{-1}{2}\left(\frac{y-b}{\sigma_2\sqrt{1-\rho^2}}\right)^2\right] dy$$

Since the integral is the pdf of r.v y with mean b and variance $\sigma_2^2(1-\rho)^2$,

$$f_1(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}, \quad \infty < x < \infty$$

Now $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx$

$$= \int_{-\infty}^{\infty} f_1(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}}\,dx$$

$$= 1$$

Note:

The marginal pdf of the r.v Y is

$$f_2(y) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{\frac{-(y-\mu_2)^2}{2\sigma_2^2}}, \quad -\infty < y < \infty.$$

Conditional pdf of a bivariate N.D:

W.K.T $f(y/x) = \frac{f(x,y)}{f(x)}$

$$= \frac{\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-1/2\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-1/2\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{\frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-1/2\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}$$

$$f(y/x) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-1/2\left(\frac{y-\mu_2}{\sigma_2}\right)^2}$$

Note:

1) conditional pdf of $Y|X$ is the normal distribution

$$N\left[\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1),\ \sigma_2^2(1-\rho^2)\right]$$

conditional mean of Y given $X=x$ is

$$E(Y|x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$$

conditional variance of Y given $X=x$ is

$$var(Y|x) = \sigma_2^2(1-\rho^2).$$

2) The conditional distribution of X given $Y=y$ is the N.D

$$N\left[\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2),\ \sigma_1^2(1-\rho^2)\right]$$

conditional. mean of X given $Y=y$ is

$$E(X|y) = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)$$

conditional var of X given $Y=y$ is

$$var(X|y) = \sigma_1^2(1-\rho^2).$$

pblms

1) If X and Y have a bivariate N.d with means μ_1 and μ_2 and positive variance σ_1^2 and σ_2^2 and correlation coefficient ρ Then P.T X and Y are independent iff $\rho=0$.

Soln: Assume X and Y are independent.

ie, $M(t_1,t_2) = M(t_1,0)\,M(0,t_2)$

$$e^{\mu_1 t_1 + \mu_2 t_2 + \frac{\sigma_1^2 t_1^2 + \sigma_2^2 t_2^2 + 2\rho\sigma_1\sigma_2 t_1 t_2}{2}} = e^{\mu_1 t_1 + \frac{\sigma_1^2 t_1^2}{2}} \cdot e^{\mu_2 t_2 + \frac{\sigma_2^2 t_2^2}{2}}$$

$$e^{\frac{2\rho\sigma_1\sigma_2 t_1 t_2}{2}} = 1.$$

$$\rho\sigma_1\sigma_2 t_1 t_2 = 0$$

$$\rho = 0$$

conversely assume $\rho = 0$

To prove x and y are independent.

ie) T.P.T: $M(t_1, t_2) = M(t_1, 0)\, M(0, t_2)$

$$M(t_1, t_2) = e^{\mu_1 t_1 + \mu_2 t_2 + \frac{\sigma_1^2 t_1^2 + \sigma_2^2 t_2^2 + 2\rho\sigma_1\sigma_2 t_1 t_2}{2}}$$

$$= e^{\mu_1 t_1 + \frac{\sigma_1^2 t_1^2}{2}} \cdot e^{\mu_2 t_2 + \frac{\sigma_2^2 t_2^2}{2}}$$

$[\because \rho = 0]$

$$= M(t_1, 0) \cdot M(0, t_2)$$

$\therefore$ x and y are independent.

2) Let x and y have a bivariate nd with the parameters $\mu_1 = 3, \mu_2 = 1, \sigma_1^2 = 16, \sigma_2^2 = 25, \rho = 3/5$. Determine the following probabilities

i) $P(3 < x < 8)$ ii) $P(3 < y < 8 \mid x = 7)$

iii) $P(-3 < y < 3)$ iv) $P(-3 < x < 3 \mid y = -4)$

soln:

i) $P(3 < x < 8) = P\left(\frac{3-\mu_1}{\sigma_1} < \frac{x-\mu_1}{\sigma_1} < \frac{8-\mu_1}{\sigma_1}\right)$

$$= P(0 < z < 5/4)$$

$$= \phi(1.25) - \phi(0)$$

$$= 0.8944 - 0.5$$

$$= 0.3944$$

ii) $P(3 < y < 8 \mid x = 7)$

$$E(y \mid x) = \mu_2 + \frac{\sigma_2}{\sigma_1}\rho(x - \mu_1)$$

$$= 1 + \frac{5}{4} \cdot 3/5\,(7 - 3)$$

$$= 4$$

$var(Y|X) = \sigma_2^2(1-\rho^2)$

$= 25(1 - 9/25)$

$= 16$ $\quad P\left(\frac{3-4}{4} < z < \frac{8-4}{4}\right)$

$\mu = 4, \sigma = 4$

$P(3 < Y < 8 | X = 7) = \phi\left(\frac{8-4}{4}\right) - \phi\left(\frac{3-4}{4}\right)$

$= \phi(1) - \phi(-0.25)$

$= \phi(1) - [1 - \phi(0.25)]$

$= 0.84 - 1 + 0.5987$

$= 0.4402$ $\quad P\left(\frac{-3-3}{4} < z < \frac{3-3}{4}\right)$

iii) $P(-3 < X < 3) = \phi(0) - \phi(-3/2)$ $\quad P\left(\frac{-6}{4} < z < 0\right)$

$= 0.5 - 1 + 0.9332$

$= 0.4332$.

iv) $E(X|Y) = \mu_1 + \frac{\sigma_1}{\sigma_2}\rho(y - \mu_2)$

$= 0.6$

$var(X|Y) = \frac{16}{5} = 3.2$

$P(-3 < X < 3 | Y = 4) = \phi\left(\frac{3-0.6}{3.2}\right) - \phi\left(\frac{-3-0.6}{3.2}\right)$

$= \phi(0.75) - \phi(-1.125)$

$= 0.7734 - 1 + 0.8686$

$= 0.642$.

3) Let X and Y have a bivariate N.D with $\mu_1 = 5$, $\mu_2 = 10$, $\sigma_1^2 = 1$, $\sigma_2^2 = 25$ and $\rho > 0$. If $P(4 < Y < 16 | X = 5) = 0.954$. Determine ρ.

Soln:

$E(Y|X=5) = \mu_2 + \frac{\sigma_2}{\sigma_1}(x - \mu_1)$

$= 10$

$var(Y|X) = \sigma_2^2(1-\rho^2)$

Gn $P(4 < Y < 16 | X = 5) = 0.954$

$$\phi\left(\frac{16-10}{\sigma}\right) - \phi\left(\frac{4-10}{\sigma}\right) = 0.954$$

$$\phi\left(\frac{6}{\sigma}\right) - 1 + \phi\left(\frac{6}{\sigma}\right) = 0.954$$

$$2\phi\left(\frac{6}{\sigma}\right) = 1.954$$

$$\phi\left(\frac{6}{\sigma}\right) = 0.977$$

From table, $\phi(2) = 0.977$ $\qquad \phi\left(\frac{6}{\sigma}\right) = \phi(2)$

$$\therefore \frac{6}{\sigma} = 2 \Rightarrow \sigma = 3$$

$$\operatorname{var}(Y|X) = \sigma_2^2(1-\rho^2)$$

$$\sigma^2 = 25(1-\rho^2)$$

$$1-\rho^2 = 9/25$$

$$\rho^2 = \frac{16}{25}$$

$$\rho = 4/5$$

Normal distribution problems:

b) point of influence of normal distribution (or) show that the graph of a pdf $N(\mu, \sigma^2)$ has point of inflection at $x = \mu - \sigma$ and $x = \mu + \sigma$.

soln:-

Condition for point of inflection of $f''(x) = 0$ and $f'''(x) \neq 0$

Give $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} \left(-\left(\frac{x-\mu}{\sigma^2}\right)\right)$$

$$f'(x) = -\left(\frac{x-\mu}{\sigma^2}\right) f(x)$$

$$f''(x) = f'(x)\left(-\left(\frac{x-\mu}{\sigma^2}\right)\right) - f(x)\left(\frac{1}{\sigma^2}\right)$$

$$= -\left[\frac{x-\mu}{\sigma^2}\right]\left(-\left(\frac{x-\mu}{\sigma^2}\right)f(x)\right) - f(x)\left(1/\sigma^2\right)$$

$$= \frac{-1}{\sigma^2}\left[-\frac{(x-\mu)^2}{\sigma^2} f(x) + f(x)\right]$$

ie) $f''(x) = 0$

$$\Rightarrow \frac{1}{\sigma^2}\left[-\frac{(x-\mu)^2}{\sigma^2} f(x) + f(x)\right] = 0$$

$$-\frac{1}{\sigma^2} f(x)\left[-\frac{(x-\mu)^2}{\sigma^2} + 1\right] = 0$$

$$-\frac{(x-\mu)^2}{\sigma^2} = -1$$

$$(x-\mu)^2 = \sigma^2$$

$$\sigma = \pm(x-\mu)$$

$$\Rightarrow \sigma = x-\mu$$

$$\sigma = x+\mu$$

$x = \sigma+\mu$ and $x = \sigma-\mu$.

7) If X is $N(\mu, \sigma^2)$. Show that

$$\mu_{2n} = \sigma^2 (2n-1)\,\mu_{2n-2}.$$

Soln:-

$$\mu_{2n} = E(X-\mu)^{2n}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^{2n} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

[$\because$ pdf of normal dist is $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$]

put $z = \dfrac{x-\mu}{\sigma}$

$$\sigma z = x-\mu$$

$$\sigma\,dz = dx.$$

Limit does not change.

$$\mu_{2n} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n} e^{-\frac{1}{2}z^2}\,\sigma\,dz$$

$$= \frac{\cancel{\sigma}}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^{2n} z^{2n} e^{-z^2/2}\,dz$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} z^{2n} e^{-z^2/2}\,dz$$

[$\because$ Integrand is an even function]

put $t = z^2/2$

$$2t = z^2 \Rightarrow z = \sqrt{2t}$$

$$2\,dt = 2z\,dz \Rightarrow \frac{dt}{z} = dz \Rightarrow \frac{dt}{\sqrt{2t}} = dz$$

when $z=0$, $t=0$

when $z=\infty$, $t=\infty$

$\therefore$ Limit does not change.

$$\mu_{2n} = \frac{\sigma^{2n}}{\sqrt{2\pi}} \, 2 \int_0^\infty (2t)^{1/2 \cdot 2n} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{\sigma^{2n}}{\sqrt{2}\cdot\sqrt{2}\sqrt{\pi}} \cdot 2 \int_0^\infty (2t)^n e^{-t} t^{-1/2} dt$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^\infty t^{n-1/2} e^{-t} dt$$

$$\mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^\infty t^{n+1/2-1} e^{-t} dt \longrightarrow (1)$$

~~put n=n-1 in (1)~~

$$\mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \Gamma(n+1/2) \longrightarrow (1)$$

$$\left[\because \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx\right]$$

put $n=n-1$ in (1), we get

$$\mu_{2(n-1)} = \frac{2^{n-1} \sigma^{2(n-1)}}{\sqrt{\pi}} \Gamma(n-1+1/2)$$

$$\mu_{2n-2} = \frac{2^{n-1} \cdot \sigma^{2n-2}}{\sqrt{\pi}} \Gamma(n-1/2) \longrightarrow (2)$$

(1) $\div$ (2), we get

$$\frac{\mu_{2n}}{\mu_{2n-2}} = \frac{\frac{2^n \sigma^{2n}}{\sqrt{\pi}} \Gamma(n+1/2)}{\frac{2^{n-1}\sigma^{2n-2}}{\sqrt{\pi}} \Gamma(n-1/2)}$$

$$= \frac{2^n \sigma^{2n} \Gamma(n+1/2)}{2^n . 2^{-1} . \sigma^{2n} . \sigma^{-2} \Gamma(n-1/2)}$$

$$= \frac{\Gamma(n+1/2+1-1)}{2^{-1} \sigma^{-2} \Gamma(n-1/2)}$$

$$= \frac{\Gamma(n-1/2+1)}{2^{-1} \sigma^{-2} \Gamma(n-1/2)} \quad [\because \Gamma(n) = (n-1)\Gamma(n-1)]$$

$$= \frac{2\sigma^2 (n-1/2) \Gamma(n-1/2)}{\Gamma(n-1/2)}$$

$$= 2\sigma^2 \left(\frac{2n-1}{2}\right)$$

$$\frac{\mu_{2n}}{\mu_{2n-2}} = \sigma^2(2n-1)$$

$$\mu_{2n} = \sigma^2(2n-1)\mu_{2n-2}$$

Hence proved.

UNIT - IV

DISTRIBUTION OF FUNCTIONS OF RANDOM VARIABLES

Unit- IV

Sampling theory

Defn : statistic

A function of one or more random variables that does not depend upon any unknown parameters is called a statistic.

Ex:

The random variable $Y = \sum_{i=1}^{n} x_i$ is statistic when $x_1, x_2, \dots, x_n$ denote 'n' random variables that have a joint pdf

$$f_1(x) = \begin{cases} p^x (1-p)^{1-x}, & x = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

The random variable $Y = \frac{X-\mu}{\sigma}$ is not statistic unless if μ and σ are known numbers.

Random sample:-

Let $x_1 x_2 x_3 \dots x_n$ denote independent random variables each of which has the same but possibly unknown pdf $f(x)$.

ie) The probability function of $x_1, x_2, \dots, x_n$ are respectively $f_1(x_1) = f(x_1)$ $f_2(x_2) = f(x_2) \dots$ $f_n(x_n) = f(x_n)$

So that the joint p.d.f is $f(x_1, x_2, \dots, x_n)$ The random variables $X_1, X_2, \dots, X_n$ are said to constitute a random sample from a distribution that has pdf $f(x)$

ie) The observation of a random sample are identically distributed.

Defn:-

Let $x_1, x_2, \ldots, x_n$ denote a sample of size n from a given distribution the statistic

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$ is called the mean as the random sample and the statistic

$s^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n}$ is called the variance of the random sample.

problem:

1) Show that $s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2$

where $\bar{x} = \sum_{i=1}^{n} x_i/n$.

soln:-

Given $s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$

To prove : $\frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2$

L.H.S

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} (x_i^2 - 2x_i\bar{x} + \bar{x}^2)$$

$$= \frac{1}{n} \left(\sum_{i=1}^{n} x_i^2 - 2\bar{x} \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \bar{x}^2 \right)$$

$$= \sum_{i=1}^{n} \frac{x_i^2}{n} - 2\bar{x}\bar{x} + \bar{x}^2$$

$$= \sum_{i=1}^{n} \frac{x_i^2}{n} - 2\bar{x}^2 + \bar{x}^2$$

$$= \sum_{i=1}^{n} \frac{x_i^2}{n} - \bar{x}^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

L.H.S = R.H.S

2) Let X_1, X_2, X_3 random sample of size 3 from a distribution $N(6,4)$ determine the probability that the largest sample item exceeds 8.

soln:

Let $Y = \max\{x_1, x_2, x_3\}$

$$P(Y>8) = P(X_1>8, X_2>8, X_3>8)$$

$$= [P(x_1>8), P(x_2>8), P(x_3>8)]$$

$$= [1-P(x_1 \le 8), 1-P(x_2 \le 8), 1-P(x_3 \le 8)]$$

$$= \left[1-\phi\left(\frac{x-\mu}{\sigma}\right), 1-\phi\left(\frac{x-\mu}{\sigma}\right), 1-\phi\left(\frac{x-\mu}{\sigma}\right)\right]$$

$$= [1-\phi(2/2), 1-\phi(2/2), 1-\phi(2/2)]$$

$$= [1-\phi(1), 1-\phi(1), 1-\phi(1)]$$

$$= [1-\phi(1)]^3$$

$$= [1-0.84]^3$$

$$P(Y>8) = 4.096 \times 10^{-3}$$

3) If $X_i = i$, $i = 1, 2, \ldots, n$ compute the value of $\bar{x}$ and s^2.

soln:

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$$

$$= \sum_{i=1}^{n} i/n$$

$$= \frac{1}{n}[1+2+\ldots+n]$$

$$= \frac{1}{n}\left[\frac{n(n+1)}{2}\right]$$

$$\bar{x} = \frac{n+1}{2}$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{(n+1)^2}{4} \quad \left[\because \bar{x} = \frac{n+1}{2}\right]$$

$$= \frac{1}{n}\left[1^2+2^2+\dots+n^2\right] - \frac{(n+1)^2}{4}$$

$$= \frac{1}{n}\frac{n(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= (n+1)\left[\frac{2n+1}{6} - \frac{n+1}{4}\right]$$

$$= (n+1)\left[\frac{2(2n+1)-3(n+1)}{12}\right]$$

$$= \frac{n+1}{12}\left[4n+2-3n-3\right]$$

$$= \frac{n+1}{12}(n-1)$$

$$s^2 = \frac{n^2-1}{12}.$$

4) Let $y_i = a+bx_i$, $i=1,2,\dots,n$ where a and b constant find $\bar{y} = \sum_{i=1}^{n}\frac{y_i}{n}$ and $s^2y = \sum_{i=1}^{n}\frac{(y_i-\bar{y})^2}{n}$ interms of a and b where $\bar{x} = \sum_{i=1}^{n}\frac{x_i}{n}$ and $s^2x = \sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{n}$.

soln:- Given $y = a+bx$

i) $\bar{y} = \sum_{i=1}^{n}\frac{y_i}{n}$

$$= \sum_{i=1}^{n}\frac{(a+bx_i)}{n}$$

$$= \sum_{i=1}^{n}\frac{a}{n} + b\sum_{i=1}^{n}\frac{x_i}{n}$$

$$= n \times \frac{a}{n} + b\bar{x} \quad \left[\because \bar{x} = \sum_{i=1}^{n}\frac{x_i}{n}\right]$$

$$\bar{y} = a + b\bar{x}$$

(ii) $$s^2y = \sum_{i=1}^{n} \frac{(y_i - \bar{y})^2}{n}$$

$$= \sum_{i=1}^{n} \frac{[(a+bx_i) - (a+b\bar{x})]^2}{n}$$

$$= \sum_{i=1}^{n} \frac{(a+bx_i - a - b\bar{x})^2}{n}$$

$$= \sum_{i=1}^{n} \frac{b^2(x_i - \bar{x})^2}{n}$$

$$= b^2 \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n}$$

$$= b^2 s^2x \quad \left[\because s^2x = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n}\right]$$

$$s^2y = b^2 s^2x.$$

Transformation of variable of the discrete type:

[Using change of variable technique]

Let X be a random variable of discrete type having pdf. f(x)

Let A denote the set of discrete points at each of which $f(x) > 0$

consider the random variable $Y = u(X)$

To determine the pdf $g(y)$ of Y, consider the transformation $y = u(x)$ to which define a 1-1 transformation that maps A onto B.

If we solve $y = u(x)$ for x interms of y say $x = w(y)$

$Y = Y[u(x) = Y]$ and $x = \omega(y)$ are equivalent.

The p.d.f of Y is

$$g(y) = P[Y = y]$$

$$= P[X = \omega(y)]$$

$$g(y) = \begin{cases} f[\omega(y)], & y \in B \\ 0 & \text{otherwise.} \end{cases}$$

Example :1

Let X on a poisson pdf

$$f(x) = \begin{cases} \dfrac{e^{-m} m^x}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Let $A = \{X : X = 0, 1, 2, \ldots\}$ where $f(x) > 0$ define a random variable $Y = 4X$. find the pdf of Y by using the change of variable techniques.

Soln:-

$Y = 4X$ is the transformation from X to Y.

Transformation $Y = 4x$ maps the space A on space B

Let $g(y)$ be a pdf of Y.

$$Y = y$$

$$4x = y$$

$$x = y/4$$

The event $Y = y$ and $X = y/4$ are equivalent.

$$g(Y) = P[Y=y]$$
$$= P[X = y/4]$$
$$g(y) = f(y/4)$$
$$g(y) = \begin{cases} \dfrac{e^{-m} m^{y/4}}{(y/4)!} & y = 0, 4, 8, \ldots \end{cases}$$

Thus we obtain the pdf of $Y = 4X$.

Ex: 2

Let X have a binomial pdf

$$f(x) = \begin{cases} \dfrac{3!}{x!(3-x)!} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{3-x} & x = 0, 1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$$

Find the pdf $g(y)$ of random variable $Y = X^2$

Soln:-

Given $f(x) = \begin{cases} \dfrac{3!}{x!(3-x)!} \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{3-x}, & x = 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$

The transformation $Y = X^2$ maps

$A = \{x : x = 0, 1, 2, 3\}$ onto to $B = \{y : y = 0, 1, 4, 9\}$

since $x = \pm\sqrt{y}$

But there are non-negative value of x in A

Take unique value

$$x = \sqrt{y}$$
$$g(y) = P[Y = y]$$
$$= P[X^2 = Y]$$
$$= P[X = \sqrt{y}]$$
$$= f(\sqrt{y})$$

$$g(y) = \begin{cases} \frac{3!}{(\sqrt{y})!\,(3-\sqrt{y})!} (2/3)^{\sqrt{y}} (1/3)^{3-\sqrt{y}} & y = 0,1,4,9 \\ 0 & \text{otherwise} \end{cases}$$

Defn:

Let $f(x_1, x_2)$ be a joint pdf of two random variables x_1, x_2 of discrete type

Let A be the set of points at which $f(x_1, x_2)$

Let $Y_1 = u_1(X_1, X_2)$

$Y_2 = u_2(X_1, X_2)$

define 1-1 transformation that maps A onto B

The joint pdf of random variable Y_1 and Y_2 is given by

$$g(y_1, y_2) = \begin{cases} f[w_1(y_1, y_2), w_2(y_1, y_2)], & y_1, y_2 \in B \\ 0 & \text{otherwise} \end{cases}$$

where $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y, y_2)$

problems

1) Let X_1 and X_2 be two independent random variable that have the poisson distribution with mean μ_1, μ_2 respectively the joint pdf of X_1 and X_2

$$\begin{cases} \frac{\mu_1^{x_1} \mu_2^{x_2} e^{-\mu_1-\mu_2}}{x_1!\, x_2!}, & x_1 = 0,1,2,\ldots \; x_2 = 0,1,2,\ldots \\ 0 & \text{elsewhere} \end{cases}$$

If A is the set of points (x_1, x_2) where each x_1 and x_2 is the non-negative integer find the pdf of $Y_1 = X_1 + X_2$ and Y_2

find the ~~pdf of Y_1~~ marginal pdf of Y_1

soln:-

$$A = \{(x_1, x_2) / x_1 = 0, 1, \ldots \quad x_2 = 0, 1, 2, \ldots\}$$

We have to find the pdf of

$$Y_1 = X_1 + X_2 \text{ and } Y_2$$

Take $y_2 = x_2$

$y_1 = x_1 + x_2$, $y_2 = x_2$ has a 1-1 transformation that maps A onto

$$B = \{(y_1, y_2) / y_1 = 0, 1, 2, \ldots \quad y_2 = 0, 1, 2, \ldots y_1\}$$

The inverse function of $x_1 = y_1 - y_2$

$x_2 = y_2$

The joint pdf of Y_1 and Y_2

$$g(y_1, y_2) = P[Y = y_1 y_2]$$
$$= P[X = x_1 x_2]$$
$$= f(x_1, x_2) = f(y_1 - y_2, y_2)$$

$$g(y_1, y_2) = \frac{e^{-\mu_1 - \mu_2} \mu_1^{y_1 - y_2} \mu_2^{x_2}}{(y_1 - y_2)! \, (y_2)!}$$

To find the marginal pdf of Y_1

Let $g_1(y_1)$ be the marginal pdf of Y_1

$$g_1(y_1) = \sum_{y_2 = 0}^{y_1} g(y_1, y_2)$$

$$= \sum_{y_2 = 0}^{y_1} \frac{e^{-\mu_1 - \mu_2} \mu_1^{y_1 - y_2} \mu_2^{y_2}}{(y_1 - y_2)! \, y_2!}$$

$$= \sum_{y_2 = 0}^{y_1} \frac{y_1! \left(e^{-\mu_1 - \mu_2} \mu_1^{y_1 - y_2} \mu_2^{y_2}\right)}{y_1! \, (y_1 - y_2)! \, y_2!}$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{y_1!}\left[\frac{y_1!\,\mu_1^{y_1-0}\mu_2^{0}}{(y_1-0)!\,0!} + \frac{(y_1)!\,\mu_1^{y_1-1}\mu_2^{1}}{(y_1-1)} + \dots + \frac{(y_1)!\,\mu_1^{y_1-y_1}\mu_2^{y_1}}{(y_1-y_1)!\,y_1!}\right]$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{y_1!}\left[\mu_1^{y_1} + \frac{y_1(y_1-1)!\,\mu_1^{y_1-1}\mu_2}{(y_1-1)!} + \dots + \mu_1^{0}\mu_2^{y_1}\right]$$

$$= \frac{e^{-(\mu_1+\mu_2)}}{y_1!}\left[\mu_1^{y_1} + y_1C_1\mu_1^{y_1-1}\mu_2^{1} + y_1C_2\mu_1^{y_1-2}\mu_2^{2} + \dots + \mu_2^{y_1}\right]$$

$$\left[\because (\mu_1+\mu_2)^n = \mu_1^n + nC_1\mu_1^{n-1} + \dots + \mu_2^n\right]$$

$$g_1(y_1) = \begin{cases} e^{-(\mu_1+\mu_2)}(\mu_1+\mu_2)^n y & , y=0,1,2\dots \\ 0 & \text{elsewhere} \end{cases}$$

$Y_1 = X_1 + X_2$ is a poisson distribution with parameter $\mu_1+\mu_2$

2) Let x have a pdf $f(x) = \begin{cases} 1/3 & x=1,2,\dots \\ 0 & \text{otherwise} \end{cases}$

Find pdf of $Y = 2x+1$

soln:- Given $f(x) = \begin{cases} 1/3 & x=1,2,\dots \\ 0 & \text{otherwise} \end{cases}$

$y = 2x+1$ has 1-1 transmitted that maps $A = \{x : x = 1,2,3\}$ onto $B = \{y : y = 3,5,7\}$

We have the single valued inverse function $x = \frac{y-1}{2}$ by change of variables technique

The pdf of Y is

$$g(y) = \begin{cases} f\left(\frac{y-1}{2}\right) & y = 3, 5, 7 \\ 0 & \text{otherwise} \end{cases}$$

$$g(y) = \begin{cases} f\left(\frac{y-1}{2}\right) & y \in B \\ 0 & \text{otherwise} \end{cases}$$

The pdf of Y is $g(y) = \begin{cases} \frac{1}{3} & y \in B \\ 0 & \text{otherwise} \end{cases}$

3) Let X_1, X_2 have the joint pdf

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{36}, & x_1 = 1, 2, 3,\ x_2 = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint pdf of $Y_1 = X_1 X_2$, $Y_2 = X_2$ and then find marginal pdf of Y_1

Soln:

Given $f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{36} & x_1 = 1, 2, 3 \quad x_2 = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$

$$A = \{(x_1, x_2),\ x_1, x_2 = 1, 2, 3\}$$

Define $Y_1 = X_1 X_2$ and $Y_2 = X_2$ has 1-1 transmittion on $y_1 = x_1 x_2$ and $y_2 = x_2$ that maps A onto $B = \left\{ (y_1, y_2)\ \begin{matrix} y_1 = 1, 2, 3, 4, 6, 9 \\ y_2 = 1, 2, 3 \end{matrix} \right\}$

The interval are $x_1 = \frac{y_1}{y_2}$, $x_2 = y_2$

Now the joint pdf of Y_1 and Y_2 is

$$g(y_1, y_2) = \begin{cases} f(y_1/y_2, y_2), & y_1, y_2 \in B \\ 0 & \text{otherwise} \end{cases}$$

$$g(y_1, y_2) = \frac{y_1 y_2}{y_2\, 36} = \frac{y_1}{36}$$

$$g(y_1, y_2) = \begin{cases} \frac{y_1}{36}, & y_1 = 1,2,3,4,6,9 \\ 0 & \text{otherwise} \end{cases}$$

The marginal pdf of Y_1 is

$$g_1(y_1) = \sum_{y_2} g(y_1, y_2)$$

$$= \begin{cases} \sum_{y_2=1}^{3} \frac{y_1}{36}, & y_1 \in B \\ 0 & \text{otherwise} \end{cases}$$

4) Let X has p.d.f $f(x)$

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^x, & x = 1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

Find the p.d.f of $Y = X^3$.

soln:- Let $A = \{x \mid x = 1,2,3\}$

Define $Y = X^3$ has 1-1 transformation that maps A onto $B = \{y \mid y = 1, 8, 27\}$

We have the single valued inverse function is $x = y^{1/3}$

By change of variable technique

The pdf of Y is

$$g(y) = f(w(y)) - f(y^{1/3})$$

$$= \left(\frac{1}{2}\right)^{1/3}$$

$$g(y) = \begin{cases} (1/2)^{y^{1/3}} & y = 1, 8, 27 \\ 0 & \text{otherwise} \end{cases}$$

Transformation variance continuous type:

Let x be a random variable of a continuous type having the pdf $f(x)$.

Let A be the one dimensional space where $f(x) > 0$

consider the random variable

$$Y = U(x)$$

To determine the $g(y)$ of Y

consider the transformation $Y = U(x)$ to which defines a one to one transformation that maps A onto B.

Let the inverse function of $U(x)$ is denoted by $x = u^{-1}(y)$

$$x = w(y)$$

Let the derivative $\frac{dx}{dy} = w'(y)$ for all points y.

when the p.d.f of random variable $Y = U(x)$ is given by

$$g(y) = \begin{cases} f(w(y)) \, |w'(y)| & , y \in B \\ 0 & \text{otherwise} \end{cases}$$

where $|w'(y)|$ represented the absolute value of $w'(y)$

Let us called $\frac{dx}{dy} = w'(y)$ us the Jacobian of the transformation is denoted by J pblm: 1 Let x be a random variable of a continuous type having the pdf $f(x)$

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

consider the r.v $y = 8x^3$ find p.d.f of Y.

soln:-

Given x be continuous r.v

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$A = \{x \mid 0 < x < 1\}$$

consider the r.v $y = 8x^3$

$y = 8x^3$ has one-one transformation

$A = \{x \mid 0 < x < 1\}$ $\quad B = \{y \mid 0 < y < 8\}$

every $x \in A$ we have

$$y = 8x^3 \in B$$

every $y \in B$

$$x = \frac{y^{1/3}}{2} \in A$$

hence inverse function is

$$x = \frac{y^{1/3}}{2} = w(y)$$

Now the Jacobian of transformation is

$$J = w'(y) = \frac{dx}{dy} = \frac{1}{2}\left(\frac{1}{3} y^{1/3 - 1}\right)$$

$$J = \frac{1}{6} y^{-2/3}$$

$$|J| = \frac{1}{6y^{2/3}}$$

The pdf of Y is

$$g(y) = f[w(y)]\,|J|$$

$$= f\left(\frac{y^{1/3}}{2}\right) \frac{1}{6y^{2/3}}$$

$$= \frac{1}{6y^{1/3}}$$

$$\therefore g(y) = \begin{cases} \frac{1}{6y^{1/3}} & 0<y<8 \\ 0 & \text{otherwise} \end{cases}$$

pblm: 2
Let x have the pdf

$$f(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } Y = -2\log x$$

Show that y has χ^2 distribution with degrees of freedom 2.

soln:

Given $f(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$$A = \{x \mid 0<x<1\}$$

The random variable $Y=-2\log x$ has 1-1 transformation that maps A onto B.

$$B = \{y \mid 0<y<\infty\}$$

The values of x are all positive in A

$$y = -2\log x, \quad 0<y<\infty$$

$$-y/2 = \log x$$

$$e^{-y/2} = x$$

Hence the inverse function $x = e^{-y/2} = w(y)$

Now the Jacobian of transformation

$$J = w'(y) = \frac{dx}{dy}$$

$$= e^{-y/2}(-1/2)$$

$$= -\frac{1}{2e^{y/2}}$$

$$\therefore \ |J| = \frac{1}{2e^{y/2}}$$

The p.d.f of y is $g(y) = f(w(y))|J|$

$$= f(e^{-y/2}) \cdot \frac{1}{2e^{y/2}}$$

$$= 1 \cdot \frac{1}{2e^{y/2}}$$

$$g(y) = \begin{cases} \frac{1}{2} e^{-y/2}, & y \in B \\ 0, & \text{otherwise} \end{cases}$$

We know that distribution of χ^2 with r degrees of freedom is

$$g(x) = \frac{1}{\Gamma(r/2)\, 2^{r/2}} x^{r/2 - 1} e^{-x/2}$$

Now $\frac{r}{2} - 1 = 0 \Rightarrow \frac{r}{2} = 1 \Rightarrow r = 2$

$$\Gamma r/2 = \Gamma(1) = 1$$

$$2^{r/2} = 2^1 = 2$$

$$\chi^2(2)$$

$\therefore$ Y has χ^2- distribution with degrees of freedom 2.

3) If the p.d.f of X is $f(x)$

$$f(x) = \begin{cases} 2xe^{-x^2} & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$ determine the p.d.f of $Y = x^2$

soln:-

Given $f(x) = \begin{cases} 2xe^{-x^2} & 0<x<\infty \\ 0 & \text{otherwise} \end{cases}$

$A = \{x \mid 0<x<\infty\}$

$y=x^2$ has 1-1 transformation that maps A onto B.

$B = \{y \mid 0<y<\infty\}$

The value of x are all positive in A

Every $x \in A$, we have $y=x^2 \in B$

Every $y \in B$, we have $x=\sqrt{y} \in A$

Hence the inverse function is

$x = \sqrt{y} = w(y)$

Now

The Jacobian transformation

$J = w'(y) = \frac{dx}{dy} = \frac{1}{2} y^{-1/2} = \frac{1}{2y^{1/2}}$

$|J| = \frac{1}{2\sqrt{y}}$

The pdf of $y=x^2$ is

$g(y) = F[w(y)]\,|J|$

$= 2\sqrt{y}\, e^{-(\sqrt{y})^2} \cdot \frac{1}{2\sqrt{y}}$

$g(y) = e^{-y}$

$g(y) = \begin{cases} e^{-y} & 0<y<\infty \\ 0 & \text{elsewhere} \end{cases}$

Let X ha the pdf $f(x) = \begin{cases} x^2/9 & 0<x<3 \\ 0 & \text{elsewhere} \end{cases}$

find all pdf of $y=x^3$

Soln:- Given $f(x) = \begin{cases} x^2/9 & 0 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$

$$A = \{x / 0 < x < 3\}$$

$y = x^3$ has 1-1 transformation that maps A onto B.

$$B = \{y / 0 < y < 27\}$$

every $x \in A$, we have $y = x^3 \in B$

every $x \in B$, we have $y^{1/3} = x \in A$

Hence the inverse function is

$$x = y^{1/3} = w(y)$$

Now Jacobian transformation is

$$J = w'(y) = \frac{dx}{dy} = \frac{1}{3} y^{-2/3}$$

$$= \frac{1}{3} y^{2/3}$$

The p.d.f of $y = x^3$ is

$$g(y) = F[w(y)]|J|$$

$$= F(y^{1/3}) \frac{1}{3y^{2/3}} = \frac{y^{2/3}}{9} = \frac{1}{3y^{2/3}}$$

$$= \frac{1}{27}$$

$$g(y) = \begin{cases} \frac{1}{27}, & 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Define the two M variables $Y_1 = \mu_1(X_1, X_2)$ $Y_2 = \mu_2(X_1, X_2)$. Find joined pdf of Y_1 and Y_2 and marginal pdf Y_1 and Y_2

proof:-

Given $Y_1 = \mu_1(x_1, x_2)$

$Y_2 = \mu_2(x_1, x_2)$

The transformation is

$$y_1 = \mu_1(x_1, x_2)$$
$$y_2 = \mu_2(x_1, x_2)$$

Define the 1-1 transformation A onto B In the y_1 and y_2 plane with non identically vanishing Jacobian

Let the inverse function $f^{(1)}$ then

$x_1 = \omega_1(y_1, y_2)$, $x_2 = (y_1, y_2)$

Let A be the subset A and let B be the subset B such that the transformation A onto B

The events $(x_1, x_2) \subset A$ and

$(y_1, y_2) \subset B$ are equivalent

Hence the $Pr[(y_1, y_2) \in B] = Pr[(x_1, x_2) \in A]$

$$= \iint_A f(x_1, x_2)\, dx_1\, dx_2$$

$$= \iint_A f[\omega_1(y_1, y_2)\, \omega_2(y_1, y_2)]\, dx_1\, dx_2$$

$$g(y_1, y_2) = f[\omega_1(y_1, y_2)\, \omega_2(y_1, y_2)]\, |J|$$

where J be Jacobian transformation which given by

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

This is true for all $B \in B$

Hence the joined pdf of y_1, y_2 is given by

$$g(y_1, y_2) = \begin{cases} f[w_1(y_1,y_2) w_2(y_1,y_2)]\,|J| & ; B \in B \\ 0 & \text{elsewhere} \end{cases}$$

The marginal p.d.f of y_1 is

$$g_1(y_1) = \int_B g(y_1, y_2)\, dy_2$$

$$g_2(y_2) = \int_B g(y_1, y_2)\, dy_1.$$

Let the random variable have the pdf

$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$ and let X_1 and X_2

the random sample from the distribution. Find the pdf of the random variable $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ and the marginal pdf of Y_1 and Y_2

Soln:-

The joined pdf of X_1 and X_2

$$f(x_1, x_2) = f(x_1) f(x_2)$$
$$= 1 \cdot 1$$
$$= 1$$

$$f(x_1, x_2) = \begin{cases} 1 & 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ are having 1-1 transformation.

$A = \{ (x_1/x_2),\ 0 < x_1 < 1,\ 0 < x_2 < 1 \}$ onto B.

The inverse function is given by

$$(x_1 + x_2) = y_1 \quad \text{—} (1)$$
$$(x_1 - x_2) = y_2 \quad \text{—} (2)$$

Add ① and ② $2x_1 = y_1 + y_2$

$$x_1 = \frac{y_1 + y_2}{2}$$

Sub x_1 in ①

$$\frac{y_1 + y_2}{2} + x_2 = y_1$$

$$x_2 = y_1 - \frac{y_1 + y_2}{2}$$

$$= \frac{2y_1 - y_1 - y_2}{2}$$

$$x_2 = \frac{y_1 - y_2}{2}$$

Now $J = \dfrac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix}$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix}$$

$$= \frac{-1}{4} - \frac{1}{4}$$

$$= \frac{-2}{4}$$

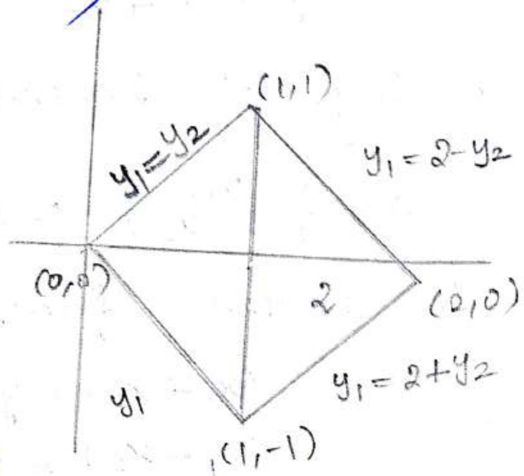
$$|J| = \frac{1}{2}$$

$$f[w_1(y), w_2(y)] \cdot |J| = g(y_1, y_2)$$

$$g(y_1, y_2) = f\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right)$$

$$= 1 \cdot \frac{1}{2}$$

$$= \frac{1}{2}$$



The p.d.f of $g(y_1, y_2)$

$$= \begin{cases} \frac{1}{2} & y \in B \\ 0 & \text{elsewhere} \end{cases}$$

The marginal pdf of $g_1(y_1) = \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_2, y_2 \in B$

$$= \int_{-y_1}^{y_1} \frac{1}{2}\, dy_2$$

$$= \frac{1}{2}(y_1 + y_1)$$

$$= \frac{1}{2} \cdot 2y_1$$

$$g(y_1) = y_1, \quad 0 \le y_1 \le 1$$

We have $x_1 = \frac{y_1 + y_2}{2}$

when $x_1 = 0$; $y_1 + y_2 = 0 \Rightarrow y_1 = -y_2$

when $x_1 = 1$; $y_1 + y_2 = 2 \Rightarrow y_1 = 2 - y_2$ and

$$x_2 = \frac{y_1 - y_1}{2}$$

when $x_2 = 0$, $y_1 - y_2 = 0 \Rightarrow y_1 = y_2$

when $x_2 = \phi$, $y_1 - y_2 = 2 \Rightarrow y_1 = 2 + y_2$

$$g_1(y_1) = \int_{y_1 - 2}^{2 - y_1} \frac{1}{2}\, dy_2, \quad 1 < y \le 2$$

$$= \frac{1}{2}[4 - 2y_1]$$

$$= \frac{1}{2} \cdot 2(2 - y_1)$$

$$= 2 - y_1, \quad 1 < y_2 \le 2$$

The marginal pdf of Y_2 is given by

$$g_2(y_2) = \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_1$$

$$= \int_{-y_{1_2}}^{2 + y_2} \frac{1}{2}\, dy_1, \quad -1 < y_2 \le 0$$

$$-g_2(y_2) = 1 + y_2, \quad 1 \le y_2 \le 0$$

$$g_2(y_2) = \int_{y_2}^{2-y_2} \frac{1}{2}\, dy_1 \qquad 0 < y_2 \le 1$$

$$= \frac{1}{2}\left[y_1\right]_{y_2}^{2-y_2}$$

$$= \frac{1}{2}\left[2 - y_2 - y_2\right]$$

$$= \frac{1}{2} \cdot 2\,(1-y_2)$$

$$g_2(y_y) = 1 - y_2$$

2) Let X_1 and X_2 be random variable size of $n=2$, from the standard normal population find the distribution $Y_1 = \frac{X_1}{X_2}$ prove that pdf of $\frac{X_1}{X_2}$ is cauchy distribution $Y_2 = X_2$.

soln:- Let us take another random variable $Y_2 = X_2$ and $A = \{(x_1, x_2),\ -\infty < x_1 < \infty$ and $-\infty < x_2 < \infty$

with $Y_1 = \frac{X_1}{X_2}$, $Y_2 = X_2$ are equivalent

The inverse function

$$x_1 = y_1 x_2$$

$$x_1 = y_1 y_2$$

$$x_2 = y_2$$

A maps onto B.

$$B = \{(y_1, y_2);\ -\infty < y < \infty,\ -\infty < y_2 < 0,\ 0 < y_2 < \infty\}$$

$$J = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$$= \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix}$$

$$|J| = y_2$$

The joined pdf $g(y_1, y_2)$

$$f\,w_1(y_1, y_2)\, f\,w_2(y_1, y_2) \cdot |J|$$

$$g(y_1, y_2) = f\,[y_1 \cdot y_2,\, y_2] \cdot y_2$$

The joined pdf of $f(x_1, x_2)$ is given by

$$f(x_1, x_2) = f(x_1)\, f(x_2)$$

Since standard normal distribution

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$= \frac{1}{2\pi} e^{-x_1^2/2 - x_2^2/2}$$

$$= \frac{1}{2\pi} e^{-1/2(x_1^2 + x_2^2)}$$

$$= \frac{1}{2\pi} e^{-1/2(y_1^2 y_2^2 + y_2^2)} \cancel{dy_2} \cdot y_2$$

$$g(y_1, y_2) = \frac{y_2}{2\pi} e^{-1/2(y_1^2 \cdot y_2^2 + y_2^2)}, \quad y_1, y_2 \in B$$

The marginal pdf of y_1 is

$$g(y_1) = 2 \int_0^\infty g(y_1, y_2)\, dy_2, \quad y_2 \in B$$

$$= 2 \int_0^\infty \frac{y_2}{2\pi} e^{-y_2^2/2 (y_1^2+1)} dy_2$$

$$= \frac{1}{\pi} \int_0^\infty y_2\, e^{-y_2^2/2 (y_1^2+1)}$$

$$u = y_2 \quad dv = e^{-y_2^2/2 (y_1^2+1)}$$

$$du = dy_2$$

$$V = \frac{e^{-y_2^2/2}(y_1^2+1)}{-\frac{2y_2}{2}(y_1^2+1)}$$

$$V = e^{-y_2^2/2}(y_1^2+1)/y_2(y_1^2+1)$$

$$g(y_1) = -\frac{1}{\pi}\left[\frac{-y_2^2/2\,(y_1^2-1)}{y_2(1+y_1^2)}\right]_0^\infty$$

β-distribution:

Let x_1 and x_2 be two independent random variable that have gamma distributions and joined pdf

$$f(x_1,x_2) = \begin{cases} \frac{1}{\Gamma\alpha\,\Gamma\beta} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2} & , 0<x_i<\infty \\ 0 & \text{elsewhere} \end{cases}$$

where $\alpha>0$, $\beta>0$.

problem:

Let $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$ we shall show that Y_1 and Y_2 are independent.

soln:-

Given that the joined pdf of x_1 and x_2 is $f(x_1,x_2) = \frac{1}{\Gamma\alpha\,\Gamma\beta} x^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2}$, $0<x_1<x_2<\infty$

$y_1 = x_1+x_2$ and $y_2 = \frac{x_1}{x_1+x_2}$ may be written as $y_2 = \frac{x_1}{y_1} \Rightarrow y_1 = x_1+x_2,\ x_1 = y_2 y_1$

$$x_2 = y_1 - y_2(x_1+x_2)$$
$$= y_1 - y_1 y_2$$
$$= y_1(1-y_2)$$

$$J = \frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$$= \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix}$$

$$= -y_1 y_2 - y_1(1-y_2)$$

$$= -y_1 y_2 - y_1 + y_1 y_2$$

$$J = -y_1$$

$$|J| = |-y_1|$$

$$|J| = y_1$$

The transformation is 1-1 and its maps

$$A = \{x_1 x_2,\ 0 < x_1 < \infty,\ 0 < x_2 < 1\}$$

onto $B = \{(y_1, y_2),\ 0 < y_1 < \infty,\ 0 < y_2 < 1\}$

$$x_1 = y_1 y_2$$

when $x_1 = 0$, $y_1 = 0$ when $x_2 = 0$, $y_2 = 1$

when $x_1 = \infty$, $y_1 = \infty$ when $x_2 = 0$, $y_2 = 1$

when $x_2 = \infty$, $y_2 = 0$ $(0 < y_2 < 1,\ 0 < y_1 < \infty)$

The pdf of y_1 and y_2 is

$$g[(y_1, y_2)] = f(y_1 y_2 \cdot y_1(1-y_2))\, y_1$$

$$f(x_1, x_2) = \frac{1}{\Gamma\alpha\,\Gamma\beta}\, x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1 - x_2},\quad \alpha, \beta > 0$$

$$= \left[\frac{1}{\Gamma\alpha\,\Gamma\beta} (y_1 y_2)^{\alpha-1} (y_1(1-y_2))^{\beta-1} \cancel{e^{-y_1 - y_2}}\, e^{-y_1 y_2 - (y_1(1-y_2))}\right] y_1$$

$$= \frac{1}{\Gamma\alpha\,\Gamma\beta}\, y_1^{\alpha-1} y_2^{\alpha-1} y_1^{\beta-1} (1-y_2)^{\beta-1} e^{-y_1 y_2 - y_1 + y_1 y_2}\, y_1$$

$$g(y_1, y_2) = \begin{cases} \frac{1}{\Gamma\alpha\,\Gamma\beta}\, y_1^{\alpha+\beta-1}\, y_2^{\alpha-1} (1-y_2)^{\beta-1} e^{-y_1} & 0<y_1<1,\ 0<y_2<1 \\ & \alpha, \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $g(y_1) = y_1^{\alpha+\beta-1} \cdot e^{-y_1}$

$$g(y_2) = \frac{y_2^{\alpha-1}(1-y_2)^{\beta-1}}{\Gamma\alpha\ \Gamma\beta}$$

ie) $g(y_1, y_2)$ is the product of two function one is the function of y_1 alone and another one is the function y_2 alone.

$\therefore y_1$ and y_2 are stochastically independent.

Now we shall find the marginal pdf of y_1 and y_2.

If y_1 and y_2 are independent then the marginal pdf of x_2 are given by

$$g(y_2) = \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_1$$

[since i) $\Gamma\alpha = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$

ii) $\beta(m,n) = \frac{\Gamma m\ \Gamma n}{m+n}$]

$$= \int_0^\infty \frac{1}{\Gamma\alpha\ \Gamma\beta}\, y_1^{\alpha+\beta-1}\, y_2^{\alpha-1} (1-y_2)^{\beta-1} e^{-y_1}\, dy_1$$

$$= \frac{1}{\Gamma\alpha\ \Gamma\beta}\, y_2^{\alpha-1} (1-y_2)^{\beta-1} \int_0^\infty y_1^{\alpha+\beta-1} e^{-y} dy$$

$$= \frac{1}{\Gamma\alpha\ \Gamma\beta}\, y_2^{\alpha-1} (1-y_2)^{\beta-1}\, \Gamma\alpha+\beta$$

$$g(y_2) = \begin{cases} \frac{\Gamma\alpha+\beta}{\Gamma\alpha\ \Gamma\beta}\, y_2^{\alpha-1} (1-y_2)^{\beta-1}, & 0<y_2<1 \\ 0 & ,\ \text{otherwise} \end{cases}$$

The pdf of Beta distribution with parameter α and β and since y_1 and y_2 are independent.

$$g(y_1, y_2) = g(y_1)\, g(y_2)$$

$$g(y_1) = \frac{g(y_1, y_2)}{g(y_2)}$$

$$= \frac{\frac{1}{\Gamma\alpha\, \Gamma\beta}\, y_2^{\alpha-1} (1-y_2)^{\beta-1}\, y_1^{\alpha+\beta-1} e^{-y_1}}{\frac{\Gamma\alpha+\beta}{\Gamma\alpha\, \Gamma\beta}\, y_2^{\alpha-1} (1-y_2)^{\beta-1}}$$

$$g(y_1) = \begin{cases} \dfrac{y_1^{\alpha+\beta-1} e^{-y}}{\Gamma\alpha+\beta}, & 0 < y_1 < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Derivations of mean and variance.

$$\text{Mean} = E(x) = E(Y_2)$$

$$= \int_0^1 y_2\, g(y_2)\, dy_2$$

$$= \int_0^1 y_2 \frac{\Gamma\alpha+\beta}{\Gamma\alpha\, \Gamma\beta}\, y_2^{\alpha-1} (1-y_2)^{\beta-1} dy_2$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\, \Gamma\beta} \int_0^1 y_2^{\alpha+1-1} (1-y_2)^{\beta-1} dy_2$$

$$\left[\because \beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx\right]$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\, \Gamma\beta}\, \beta(\alpha+1, \beta)$$

$$\left[\because \beta(m, n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}\right]$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\, \Gamma\beta} \cdot \frac{\Gamma\alpha+1\, \Gamma\beta}{\Gamma\alpha+\beta-1} \quad \left[\because \Gamma n = n-1\, \Gamma n-1\right]$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta} \cdot \frac{\alpha\,\Gamma\alpha\;\Gamma\beta}{\alpha+\beta\;\Gamma\alpha+\beta}$$

$$\text{Mean} = \frac{\alpha}{\alpha+\beta}$$

$$\text{Variance } \sigma^2 = E(x^2) - [E(x)]^2$$

$$E(y_2^2) = \int_0^1 y_2^2\, g(y_2)\, dy_2$$

$$= \int_0^1 y_2^2 \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta}\, y_2^{\alpha-1}(1-y_2)^{\beta-1}\, dy_2$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta} \int_0^1 y_2^{\alpha+1}(1-y_2)^{\beta-1}\, dy_2$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta} \int_0^1 y_2^{\alpha+1+1-1}(1-y_2)^{\beta-1}\, dy_2$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta}\, \beta(\alpha+2, \beta)$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta} \frac{\Gamma\alpha+2\;\Gamma\beta}{\Gamma\alpha+\beta+2}$$

$$= \cancel{\Gamma\alpha+\beta \;\; \alpha+1\;\Gamma\alpha+1\;\Gamma\beta}$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha\,\Gamma\beta} \cdot \frac{\alpha+1\;\Gamma\alpha+1\;\;\Gamma\beta}{(\alpha+\beta+1)\,\Gamma\alpha+\beta+1}$$

$$= \frac{\Gamma\alpha+\beta}{\Gamma\alpha} \cdot \frac{(\alpha+1)\,\alpha\,\Gamma\alpha}{(\alpha+\beta+1)\,\alpha+\beta\;\Gamma\alpha+\beta}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)\,\alpha+\beta}$$

$$\text{variance} = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha^2+\alpha\beta+\alpha+\beta) - \alpha^3 - \alpha^2\beta - \alpha^2}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$= \frac{\alpha^3+\alpha^2\beta+\alpha^2+\alpha\beta - \alpha^3 - \alpha^2\beta - \alpha^2}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$\text{variance} = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

pblm:1 $f(x) = \begin{cases} cx(1-x)^3 & , 0 < x < 1 \\ 0 & , \text{otherwise} \end{cases}$ is

Beta distribution determine the constant.

soln:-

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 c\,x(1-x)^3\,dx = 1$$

$$c\int_0^1 x(1-x)^3\,dx = 1$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$c\int_0^1 x^{2-1}(1-x)^{4-1}\,dx = 1$$

$$c\,\beta(2,4) = 1$$

$$c\,\frac{\Gamma 2\,\Gamma 4}{\Gamma 6} = 1$$

$$c\left(\frac{1!\,3!}{5!}\right) = 1$$

$$c = \frac{5!}{3!}$$

$$= \frac{5\times4\times3\times2\times1}{3\times2\times1}$$

$$c = 20.$$

t-distribution

Let w denote a random variable that is $N(0,1)$

Let V denote a random variable that is $\chi^2(r)$ and

w and v be independent then the join pdf of w and v say $h(w,v)$ is the product of the w and that of v or

$$h(w,v) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} \frac{1}{\Gamma\frac{r}{2}\, 2^{r/2}} v^{r/2-1} e^{-v/2} & -\infty < w < \infty, 0 < v < \infty \\ 0 & \text{otherwise} \end{cases}$$

Define a new random variable T by writting

$$T = \frac{w}{\sqrt{V/r}}$$

The change of variable technique will be used to obtain the pdf $g_1(t)$ of T.

The eqns

$$t = \frac{w}{\sqrt{V/r}} \text{ and } u = v$$

define transformation that maps

$$S = \{(w,v)\ -\infty < w < \infty,\ 0 < v < \infty\}$$

one to one and onto

$T = \{(t,u),\ -\infty < t < \infty,\ 0 < u < \infty$

Since $\quad w = t \cdot \sqrt{\frac{v}{r}}$

$$= t \cdot \frac{\sqrt{u}}{\sqrt{r}}$$

Then $\quad J = \dfrac{\partial(w,v)}{\partial(t,u)} = \begin{vmatrix} \frac{\partial w}{\partial t} & \frac{\partial w}{\partial u} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial u} \end{vmatrix}$

$$= \begin{vmatrix} \frac{\sqrt{u}}{\sqrt{r}} & \frac{t}{2\sqrt{u}\sqrt{r}} \\ 0 & 1 \end{vmatrix}$$

$$|J| = \frac{\sqrt{u}}{\sqrt{r}}$$

The join pdf of t and u is given by

$$g(t,u) = f(w,v)\ |J|$$

$$g(t,u) = f\left(t\frac{\sqrt{u}}{\sqrt{r}},\ u\right)|J|$$

$$g(t,u) = \int_0^\infty \left(\frac{1}{\sqrt{2\pi}}\ e^{-t^2\frac{u}{2r}}\ \frac{1}{\Gamma\frac{r}{2}\ 2^{r/2}}\ u^{r/2-1}\ e^{-u/2}\right)\frac{\sqrt{u}}{\sqrt{r}}du$$

$$= \int_0^\infty \left(\frac{1}{\sqrt{2\pi}\,\Gamma r/2\ 2^{r/2}}\ e^{-t^2\frac{u}{2r}}\ u^{r/2-1}\ e^{-u/2}\right)\frac{\sqrt{u}}{\sqrt{r}}du$$

$$|t| < \infty,\quad 0 < u < \infty$$

The marginal pdf of T is the

$$g_1(t) = \int_{-\infty}^{\infty} g(t,u)\,du$$

$$g_1(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi r}\,\Gamma r/2\ 2^{r/2}}\ u^{\left(\frac{r+1}{2}\right)-1}\ e^{\left[\frac{-u}{2}\left(1+\frac{t^2}{r}\right)\right]}du$$

Let $\quad z = \left[-\frac{u}{2}\left(1+\frac{t^2}{r}\right)\right]$

$$u = \frac{2z}{1+\frac{t^2}{r}}$$

$$du = \frac{2}{1+\frac{t^2}{r}}\, dz$$

$$= \int_0^\infty \frac{1}{\sqrt{2}\,\sqrt{\pi r}\,\Gamma r/2\, 2^{r/2}} \left(\frac{2z}{1+t^2/r}\right)^{\left(\frac{r+1}{2}\right)-1} e^{-z}\, \frac{2}{1+t^2/r}\, dz$$

$$= \int_0^\infty \frac{1}{2^{1/2}\sqrt{\pi r}\,\Gamma r/2\, 2^{r/2}}\, z^{\left(\frac{r+1}{2}\right)-1} \left(\frac{2}{1+t^2/r}\right)^{\frac{r+1}{2}} e^{-z}\, dz$$

$$= \int_0^\infty \frac{1}{2^{\frac{r+1}{2}}\sqrt{\pi r}\,\Gamma r/2}\, z^{\left(\frac{r+1}{2}\right)-1}\, \frac{2^{\frac{r+1}{2}}}{\left(1+t^2/r\right)^{\left(\frac{r+1}{2}\right)}}\, e^{-z}\, dz$$

$$= \int_0^\infty \frac{z^{\left(\frac{r+1}{2}\right)-1}\, e^{-z}}{\Gamma r/2\, \sqrt{\pi r}\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}}\, dz$$

$$\because \Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx.$$

$$g_1(t) = \frac{\Gamma r + 1/2}{\sqrt{\pi r}\,\Gamma r/2\, \left(1+t^2/r\right)^{\frac{r+1}{2}}}$$

F-distribution:

consider two independent chi-square random variable u and v having r_1 and r_2 degree of freedom respectively the join pdf $h(u,v)$ of u and v is then

$$h(u,v) = \begin{cases} \dfrac{1}{\Gamma(r_1/2)\Gamma(r_2/2)\,2^{(r_1+r_2)}} u^{\frac{r_1}{2}-1} v^{\frac{r_2}{2}-1} e^{-\frac{(u+v)}{2}} \\ 0 \quad \text{elsewhere} \end{cases}$$

We define the new random variable

$$W = \frac{u/r_1}{V/r_2}$$

and we propose finding the p.d.f $g_1(\omega)$ of ω

The eqn

$$\omega = \frac{u/r_1}{V/r_2}, \quad z = V$$

define a 1-1 transformation that maps set $S = \{(u,v)\ 0<u<\infty,\ 0<v<\infty\}$

onto the set

$$T = \{(\omega, v)\ 0<\omega<\infty,\ 0<z<\infty\}$$

Since $u = \left(\frac{r_1}{r_2}\right) z w$

$v = z$

Jacobian of the transformation

$$|J| = \frac{\partial(u,v)}{\partial(w,z)} = \begin{vmatrix} \frac{\partial u}{\partial w} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial w} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{r_1}{r_2} z & \frac{r_1}{r_2} w \\ 0 & 1 \end{vmatrix}$$

$$|J| = \left(\frac{r_1}{r_2}\right) z$$

The join pdf $g(w,z)$ of the random variable w and $z = v$ is

$$g(w,z) = \frac{1}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{(r_1+r_2)/2}} \left(\frac{r_1 z w}{r_2}\right)^{\frac{r_1}{2}-2} z^{r_2 - \frac{2}{2}} e^{\left(-\frac{z}{2}\left(\frac{r_1 w}{r_2}+1\right)\right)} \frac{r_1}{r_2} z$$

provided that $(w,z) \in T$ and zero elsewhere

The marginal p.d.f $g_1(w)$ of w is then

$$g_1(w) = \int_{-\infty}^{\infty} g(w,z)\,dz$$

$$= \int_0^{\infty} \frac{\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} (w)^{\frac{r_1-2}{2}}}{\Gamma\frac{r_1}{2}\Gamma\frac{r_2}{2}\, 2^{\left(\frac{r_1+r_2}{2}\right)}} z^{\frac{(r_1+r_2)}{2}-1} e^{\left[-\frac{z}{2}\left(\frac{r_1 w}{r_2}+1\right)\right]} dz$$

If we change the variable of Integration by writting

$$y = \frac{z}{2}\left(\frac{r_1}{r_2}w + 1\right) \Rightarrow dy = \frac{dz}{2}\left(\frac{r_1 w}{r_2} + 1\right)$$

$$\Rightarrow z = 2y\left(\frac{r_2}{r_1 w} + 1\right), \quad dz = 2\,dy\left(\frac{r_2}{r_1 w + r_2}\right)$$

it can be see that

$$g_1(w) = \int_0^{\infty} \frac{\left(\frac{r_1}{r_2}\right)^{r_1/2} w^{\frac{r_1}{2}-1}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{\left(\frac{r_1+r_2}{2}\right)}} \left(\frac{2y}{\frac{r_1 w}{r_2}+1}\right)^{\frac{r_1+r_2}{2}-1}$$

$$e^{-y}\left(\frac{2}{\frac{r_1 w}{r_2}+1}\right) dy$$

$$= \begin{cases} \dfrac{\Gamma\left(\frac{r_1+r_2}{2}\right)\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}}}{\Gamma\frac{r_1}{2}\,\Gamma\frac{r_2}{2}} \dfrac{w^{\frac{r_1}{2}-1}}{\left(1+\frac{r_1 w}{r_2}\right)^{\left(\frac{r_1+r_2}{2}\right)}}, & 0 < w < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Thm: Let X_1, X_2, X_3 denote a random sample from positive distribution having p.d.f

$$f(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$ show that

$$Y_1 = \frac{X_1}{X_1+X_2}, \quad Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}, \quad Y_3 = X_1+X_2+X_3$$

are mutually independent and also find the marginal p.d.f

proof:-

Given $$f(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

X_1, X_2, X_3 are mutually independent.

$$f(x_1, x_2, x_3) = f(x_1)\, f(x_2)\, f(x_3)$$
$$= e^{-x_1} e^{-x_2} e^{-x_3}$$
$$= e^{-(x_1+x_2+x_3)}$$

The transformation is $Y_1 = \frac{X_1}{X_1+X_2}$, $Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}$

$Y_3 = X_1 + X_2 + X_3$

$$Y_1 = \frac{x_1}{x_1\left(1+\frac{x_2}{x_1}\right)} \qquad Y_2 = \frac{x_1+x_2}{x_1+x_2\left(1+\frac{x_3}{x_1+x_2}\right)}$$

$$Y_3 = x_1 + x_2 + x_3$$

Then the inverse function is $y_2 = \frac{x_1+x_2}{y_3}$

$$y_2 \cdot y_3 = x_1 + x_2 \Rightarrow x_2 = y_2 \cdot y_3 - x_1$$

$$x_2 = y_2 y_3 - y_1 y_2 y_3$$

$$y_1 = \frac{x_1}{x_1+x_2} = \frac{x_1}{y_2 y_3} \Rightarrow x_1 = y_1 y_2 y_3$$

$$x_2 = y_2 y_3 - y_1 y_2 y_3$$

$$\Rightarrow x_2 = y_2 y_3 (1 - y_1)$$

$$y_3 = x_1 + x_2 + x_3$$

$$\Rightarrow x_3 = y_3 - x_1 - x_2$$

$$x_3 = y_3 - y_1 y_2 y_3 - y_2 y_3 (1 - y_1)$$
$$= y_3 - y_1 y_2 y_3 - y_2 y_3 + y_1 y_2 y_3$$
$$= y_3 - y_2 y_3$$
$$= y_3 (1 - y_2)$$

The transformation is 1-1 that maps

$$A = \{(x_1, x_2, x_3) \,/\, 0 < x_1 < \infty,\ 0 < x_2 < \infty,\ 0 < x_3 < \infty\}$$

onto

$$B = \{(y_1, y_2, y_3) \,/\, 0 < y_1 < 1,\ 0 < y_2 < 1,\ 0 < y_3 < \infty\}$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix}$$

$$= \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ y_2 y_3 & y_3(1-y_1) & y_2(1-y_1) \\ 0 & -y_3 & 1-y_2 \end{vmatrix}$$

$$= y_2 y_3 \left[y_3(1-y_1)(1-y_2) + y_2 y_3 (1-y_1) \right]$$
$$- y_1 y_3 \left[-y_2 y_3 (1-y_2) \right] + y_1 y_2 \left[y_2 y_3^2 \right]$$

$$|J| = y_2 y_3^2$$

The join pdf of Y_1, Y_2, Y_3 is

$$g(y_1, y_2, y_3) = \begin{cases} f[w_1(y_1,y_2,y_3), w_2(y_1,y_2,y_2), \\ \quad w_3(y_1,y_2,y_3)] & y_1, y_2, y_3 \in Y \\ 0 & \text{otherwise} \end{cases}$$

$$= f(x_1, x_2, x_3)$$

$$= f(y_1 y_2 y_3, y_2 y_3(1-y_1), y_3(1-y_2)) y_2 y_3^2$$

$$= e^{-(y_1 y_2 y_3 + y_2 y_3 (1-y_1) + y_3(1-y_2))} y_2 y_3^2$$

$$g(y_1, y_2, y_3) = e^{-y_3} y_2 y_3^2$$

Marginal pdf

$$g(y_1) = \int_0^\infty \int_0^1 y_2 y_3^2 e^{-y_3} \, dy_2 \, dy_3$$

$$= \int_0^\infty y_3^2 e^{-y_3} \left[\frac{y_2^2}{2} \right]_0^1 dy_3 - \int_0^\infty y_3^2 e^{-y_3} (1/2) dy_3$$

$$= \frac{1}{2} \int_0^\infty y_3^2 e^{-y_3} dy_3$$

put $u = y_3^2$ $\qquad dv = e^{-y_3} dy_3$

$du = 2y_3 dy_3$ $\qquad v = -e^{-y_3}$

$$g_1(y_1) = \frac{1}{2}\left\{\left[-y_3^2 e^{-y_3}\right]_0^{\infty} + \int_0^{\infty} 2e^{-y_3} y_3 dy_3\right\}$$

$$= \frac{1}{2}\left\{0 + 2\int_0^{\infty} e^{-y_3} y_3 dy_3\right\}$$

$$= \int_0^{\infty} e^{-y_3} y_3 dy_3$$

$u = y_3$ $\qquad dv = e^{-y_3} dy_3$

$du = dy_3$ $\qquad v = -e^{-y_3}$

$$g_1(y_1) = \left[-e^{-y_3} y_3\right]_0^{\infty} + \int_0^{\infty} e^{-y_3} dy_3$$

$$= 0 + \int_0^{\infty} e^{-y_3} dy_3$$

$$= \left[e^{-y_3}\right]_0^{\infty}$$

$$= 0 + e^0 = 1$$

$$g_2(y_2) = \int_0^{\infty}\int_0^{1} e^{-y_3} y_2 y_3^2 dy_1 dy_3$$

$$= \int_0^{\infty} e^{-y_3} y_2 y_3^2 [y_1]_0^1 dy_3$$

$$= \int_0^{\infty} e^{-y_3} y_2 y_3^2 dy_3$$

$$= \int_0^{\infty} y_2 y_3^2 e^{-y_3} dy_3$$

$$= y_2 \int_0^{\infty} y_3^2 e^{-y_3} dy_3$$

$u = y_3^2$ $\qquad dv = e^{-y_3} dy_3$

$du = 2y_3 dy_3$ $\qquad v = -e^{-y_3}$

$$g_2(y_2) = y_2 \left\{ \left[-y_3^2 e^{-y_3}\right]_0^\infty + \int_0^\infty 2e^{-y_3} y_3\, dy_3 \right\}$$

$$= y_2(0+2(1))$$

$$g_2(y_2) = 2y,$$

$$g_3(y_3) = \int_0^1 \int_0^1 e^{-y_3} y_2 y_3^2\, dy_1\, dy_2$$

$$= \int_0^1 [y_1]_0^1\, e^{-y_3} y_3^2 y_2\, dy_2$$

$$= \int_0^1 e^{-y_3} y_3^2 y_2\, dy_2$$

$$= e^{-y_3} y_3^2 \left[y_2^2/2 \right]_0^1$$

$$= e^{-y_3} y_3^2 \left[1/2 \right]$$

$$g_3(y_3) = \frac{1}{2} e^{-y_3} y_3^2$$

from the joint p.d.f of Y_1, Y_2, Y_3 are have $g(y_1, y_2, y_3) = g(y_1)\, g(y_2)\, g(y_3)$

$$e^{-y_3} y_2 y_3^2 = (1)(2y_2) \frac{1}{2} e^{-y_3} y_3^2$$

$$e^{-y_2} y_2 y_3^2 = e^{-y_3} y_2 y_3^2$$

$\therefore$ Y_1, Y_2, Y_3 are mutually independent.

1) If $f(x) = \begin{cases} \frac{1}{2} ; & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$ is the p.d.f of variable x find the p.d.f of x^2.

soln:-

Let $f(x) = \begin{cases} \frac{1}{2} & -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

Let $Y = x^2$ maps $A = \{x : -1 < x < 1\}$ onto

$B = \{ y : 0 < y < 1 \}$ and the transformation 1-1.

The inverse function is $x = \sqrt{y}$

$$J = \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

The p.d.f of $g(y) = [f(w)] \, |J|$

ie) The p.d.f $Y = x^2$ is $g(y) = f(\sqrt{y}) \cdot |J|, \; y \in B$

$$= \frac{1}{2} \cdot \frac{1}{2\sqrt{y}}$$

$$g(y) = \begin{cases} \frac{1}{4\sqrt{y}}, & y \in B \\ 0 & \text{elsewhere} \end{cases}$$

2) If x_1 and x_2 is a random variable from standard normal distribution find the joined p.d.f $Y_1 = X_1^2 + X_2^2$ and $Y_2 = X_2$ and the marginal p.d.f of Y_1.

soln:-

Let X_1 and X_2 is a random sample from a standard normal distribution.

The p.d.f of X_1 and X_2 is given by

$$f(x_1, x_2) = f(x_1) \, f(x_2)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2}$$

$$= \begin{cases} \frac{1}{2\pi} e^{-\left(\frac{x_1^2 + x_2^2}{2}\right)}, & -\infty < x_1, x_2 < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Let $Y_1 = X_1^2 + X_2^2$ and $Y_2 = X_2$ the transformation is 1-1 that maps

$A = \{(x_1, x_2) ; -\infty < x_1 < x_2 < \infty\}$ onto

$B = \{(y_1, y_2) ; 0 < y_1 < \infty, -\sqrt{y} < y_2 < \sqrt{y}\}$

$$y = \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2}(y_1 - y_2^2)^{-1/2} & \frac{1}{2}(y_1 - y_2^2)^{-1/2}(-2y_2) \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(y_1 - y_2^2)^{-1/2}$$

$$= \frac{1}{2\sqrt{y_1 - y_2^2}}$$

The joined pdf of $Y_1 Y_2$ is

$$g(y_1, y_2) = f[w_1(y_1, y_2), w_2(y_1, y_2)] |J| ; \quad y_1, y_2 \in B$$

$$g(y_1, y_2) = f[\sqrt{y_1 - y_2^2}, y_2] |J|$$

$$= \frac{1}{2\pi} e^{-\frac{(y_1 - y_2^2 + y_2^2)}{2}} \cdot \frac{1}{2\sqrt{y_1 - y_2^2}}$$

$$g(y_1, y_2) = \begin{cases} \frac{1}{4\pi\sqrt{y_1 - y_2^2}} e^{-y_1/2} & y_1, y_2 \in B \\ 0 & \text{otherwise} \end{cases}$$

The marginal pdf of Y_1 is

$$g_1(y_1) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{e^{-y_1/2}}{4\pi\sqrt{y_1 - y_2^2}} dy_2$$

$$= e^{-y_1/2} \int_{-\sqrt{y}}^{\sqrt{y}} (y_1 - y_2^2)^{-1/2} dy_2$$

$$= \frac{e^{-y_1/2}}{4\pi}\left[\frac{(y_1 - y_2^2)^{1/2}}{1/2\,(-2y_2)}\right]_{-\sqrt{y}}^{\sqrt{y}}$$

$$= \frac{e^{-y_1/2}}{4\pi}\left[\frac{(y_1 - y)^{1/2}}{-\sqrt{y}} - \frac{(y_1 - y)^{1/2}}{\sqrt{y}}\right]$$

$$= \frac{e^{-y_1/2}}{4\pi}\left[\frac{-(y_1 - y)^{1/2}}{\sqrt{y}} - \frac{(y_1 - y)^{1/2}}{\sqrt{y}}\right]$$

$$= \frac{e^{-y_1/2}}{4\pi}\left[\frac{-2(y_1 - y)^{1/2}}{\sqrt{y}}\right]$$

$$= -\frac{e^{-y_1/2}}{2\pi}\left[\frac{(y_1 - y)^{1/2}}{\sqrt{y}}\right]$$

$$g_1(y_1) = \begin{cases} -\frac{e^{-y_1/2}}{2\pi}\left(\frac{\sqrt{y_1 - y}}{\sqrt{y}}\right) & y_1, y_2 \in B \\ 0 & \text{elsewhere} \end{cases}$$

Let $x_1, x_2, \ldots$ are independent $M_i(t), i = 1, 2, \ldots, n$ Then the m.g.f of $Y = \sum_{i=1}^{n} a_i x_i$ where $a_1, a_2, \ldots, a_n$ are real constant (or)

P.T $M_y(t) = \prod_{i=1}^{n} M_i(a_i t)$

soln:-

M.g.f of Y is $M_y(t) = E(e^{ty})$

$$= E\left(e^{t\sum_{i=1}^{n} a_i x_i}\right)$$

$$= E\left(e^{ta_1x_1 + ta_2x_2 + \ldots + ta_nx_n}\right)$$

$$= E(e^{ta_1x_1})E(e^{ta_2x_2})\ldots E(e^{ta_nx_n})$$

$$E(e^{tx_i}) = M_{x_i}(t) = M_i(t)$$

$E(e^{a_i t x_i}) = M_i(a_i t)$

Mgf of $Y = M_y(t) = E(e^{ty})$

$= M_1(a_1 t)\, M_2(a_2 t) \cdots M_n(a_n t)$

$= \prod_{i=1}^{n} M_i(a_i t)$

If $x_1, x_2, \ldots, x_n$ are observation of random variable sample from a distribution with $M(t)$ then i) Mgf of $Y = \sum_{i=1}^{n} x_i$ is $M_y(t) = \prod_{i=1}^{n} M(t)$ ii) Mgf of $\bar{x} = \sum_{i=1}^{n} (1/n)\, x_i$ is $M_{\bar{x}}(t) = \sum_{i=1}^{n} M(t/n) = [M(t/n)]^n$.

soln:-

Mgf of $Y = \sum_{i=1}^{n} a_i x_i$

i) $M_Y(t) = \prod_{i=1}^{n} M_i(a_i t)$

put $a_i = 1$, $i = 1, 2, \ldots, n$

Mgf of $Y = \prod_{i=1}^{n} M_i(t) = M_1(t) M_2(t) \cdots M_n(t)$

$= M(t) M(t) \cdots M(t)$

[$x_1, x_2, \ldots, x_n$ are having mgf of $M(t)$]

ie) Mgf of $x_1 = M(t)$

M.g.f of $x_2 = M(t)$

$\vdots$

M.gf of $x_n = M(t)$

$M_y(t) = [M(t)]^n$

ii) The m.g.f of $\bar{x} = \sum_{i=1}^{n} y_n x_i$ is given by

put $a_i = y_n$, $i = 1, 2, \ldots, n$ in ①

$$M_y(t) = M(t) = \prod_{i=1}^{n} M_i(t/n)$$

$$= M_1(t/n)\, M_2(t/n) \ldots M_n(t/n)$$

$$M_y(t) = [M(t/n)]^n$$

The moment generating function technique.

Let the independent random variable x_1 and x_2 have the same p.d.f

$$f(x) = \begin{cases} x/6 & x = 1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$$ To find

i) $Pr(X_1 = 2, X_2 = 3)$ ii) $Pr(X_1 + X_2 = 3)$

iii) Find the p.d.f of $Y = x_1 + x_2$

soln:-

Given $f(x) = \begin{cases} x/6 & x = 1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$

The join p.d.f of x_1 and x_2 is

$$f(x_1, x_2) = f(x_1)\, f(x_2)$$

$$= \frac{x_1}{6} \cdot \frac{x_2}{6}$$

$$= \frac{x_1 x_2}{36}$$

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{36} & x_1 = 1, 2, 3 \quad x_2 = 1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$$

i) To find $Pr(X_1 = 2, X_2 = 3)$

$$Pr(X_1 = 2, X_2 = 3) = \frac{2 \times 3}{36} = \frac{6}{36} = 1/6.$$

ii) $Pr(X_1+X_2=3)$

The event $x_1+x_2=3$ is the union exclusive of the events with probability zero of the two mutually exclusive events

$(X_1=1, X_2=2)$ (or) $(X_1=2, X_2=1)$

$$Pr(X_1+X_2=3) = Pr(X_1=1, X_2=2) + Pr(X_1=2, X_2=1)$$

$$= \frac{1\times 2}{36} + \frac{2\times 1}{36}$$

$$= \frac{4}{36}$$

$$= \frac{1}{9}$$

iii) To find p.d.f of $Y = X_1+X_2$

Here $X_1 = 1, 2, 3$ $\quad X_2 = 1, 2, 3$

$Y_1 = 2, 3, 4, 5, 6$

Let $g(y) = Pr(X_1+X_2=y)$

i) $Y=2$ Then $Pr(X_1+X_2=2) = Pr(X_1=1, X_2=1)$

$$= \frac{1}{36}$$

ii) If $Y=3$

$$Pr(X_1+X_2=3) = Pr(X_1=1, X_2=2) + Pr(X_1=2, X_2=1)$$

$$= 4/36$$

iii) If $Y=4$ then

$$Pr(X_1+X_2=4) = Pr(X_1=1, X_2=3) + Pr(X_1=3, X_2=1) + Pr(X_1=2, X_2=2)$$

$$= \frac{3}{36} + \frac{3}{36} + \frac{4}{36}$$

$$= \frac{10}{36} = 5/18$$

iv) If $y=5$ then

$$Pr(X_1+X_2=5) = Pr(X_1=3, X_2=2) + Pr(X_1=2, X_2=3)$$

$$= \frac{6}{36} + \frac{6}{36}$$

$$= \frac{12}{36}$$

$$= \frac{1}{3}$$

v) If $Y=6$ then

$$Pr(X_1+X_2=6) = Pr(X_1=3, X_2=3)$$

$$= 9/36$$

$$= 1/4$$

The p.d.f table is

Y	2	3	4	5	6
$g(Y)$	$1/36$	$4/36$	$5/18$	$1/3$	$1/4$

Now the Mgf of Y

$$M(t) = E(e^{t(x_1+x_2)}) = E(e^{tx_1} \cdot e^{tx_2})$$

$$= E(e^{tx_1})E(e^{tx_2})$$

Since x_1 and x_2 have the same p.d.f they have the same distribution

$$E(e^{tx_1}) = E(e^{tx_2})$$

$$= \sum_{x_2=1}^{3} e^{tx_2} \frac{x_2}{6}$$

$$E(e^{tx_1}) = \frac{1}{6}e^{t} + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$$

$$M(t) = \left(\frac{1}{6}e^{t} + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}\right)^2$$

$M(t) = \frac{1}{36} e^{2t} + \frac{4}{36} e^{3t} + \frac{10}{36} e^{4t} + \frac{12}{36} e^{5t} + \frac{9}{36} e^{6t}$

$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2xz$

Let x_1 and x_2 be independent with normal distribution $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively define the random variable Y be $Y = x_1 - x_2$ find p.d.f $g(y)$ of Y.

The mgf of Y is $M(t) = E(e^{tY})$

$$= E(e^{t(x_1 - x_2)})$$

$$= E(e^{tx_1 - tx_2})$$

$$= E(e^{tx_1}).E(e^{-tx_2})$$

N.K.T $E(e^{tx_1}) = \exp(\mu_1 t + \frac{\sigma_1^2 t^2}{2})$ and

$$E(e^{-tx_2}) = \exp\left(\mu_2(-t) + \frac{\sigma_2^2(-t^2)}{2}\right)$$

$$= \exp\left(-\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right)$$

$$M(t) = E(e^{tx_1}).E(e^{-tx_2})$$

$$= \exp\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right) \exp\left(-\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right)$$

$$M(t) = \exp\left((\mu_1 - \mu_2) + \left(\frac{\sigma_1^2 + \sigma_2^2}{2}\right) t^2\right)$$

Hence Y has the pdf $g(y)$ which is $N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

Let $x_1, x_2, \dots, x_n$ be independent random variable having respectively. The normal distribution $N(\mu_1, \sigma_1^2) N(\mu_2, \sigma_2^2) \dots N(\mu_n, \sigma_n^2)$ The random variable $Y = k_1 x_1 + k_2 x_2 + \dots k_n x_n$ where $k_1, k_2, \dots, k_n$ are real constant is normally distributed with mean $k_1\mu_1+k_2\mu_2+\dots+k_n\mu_n$ and variance $k_1\sigma_1^2+k_2\sigma_2^2+\dots+k_n\sigma_n^2$ that μ is $N\left(\sum_{i=1}^{n} k_i\mu_i \sum_{i=1}^{n} k_i^2\sigma_i^2\right)$

soln:

Let $x_1, x_2, \dots, x_n$ are independent.

The m.g.f of y is

$$M(t) = E\left(\exp\left(t\left(k_1x_1+k_2x_2+\dots+k_nx_n\right)\right)\right)$$

$$= E\left(e^{tk_1x_1}\right)E\left(e^{tk_2x_2}\right)\dots E\left(e^{tk_nx_n}\right)$$

$$E\left(e^{tx_i}\right) = \exp\left(\mu_i t+\sigma_1^2 t^2/2\right)$$

for all real $i=1,2,\dots$

Hence we have $E\left(e^{tx_ik_i}\right) = \exp\left(k_i\mu_i t+\frac{k_i^2\sigma_i^2t^2}{2}\right)$

The mgf of Y is

$$M(t) = \sum_{i=1}^{n} \exp\left[\left(k_1\mu_1 t+\sigma_1^2k_i^2t^2/2\right)+\dots + \exp k_n\mu_n t+\frac{k_n^2\sigma_n^2}{2}\right)t^2\right]$$

$$M(t) = \exp\left[\sum_{i=1}^{n} k_i\mu_i t+\sum_{i=1}^{n} \frac{k_i^2\sigma_i^2}{2}t^2\right]$$

But this is the m.g.f of a distribution that is $N\left(\sum_{i=1}^{n} k_i\mu_i, \sum_{i=1}^{n} k_i^2\sigma_i^2\right)$

Thm:

Let $x_1, x_2, \dots, x_n$ denote the outcomes of Bernouli trails the M.g.f of $x_i = 1,2,\dots,n$ is $M(t) = (1-p+p_2e^t)$ P.T $Y = \sum_{i=1}^{n} x_i$ is $b(n,p)$

Soln:

Here given that mgf x_i is

$$M(t) = (1-p+p_2 e^t)$$

$$\text{Mgf of } Y = \sum_{i=1}^{n} x_i = [M(t)]^n$$

$$\text{M.g.f of } y = (1-p+p_2 e^t)^n = (q+p_2 e^t)^n$$

y is $b(n,p)$.

Let $x_1, x_2, \ldots, x_n$ be independent random variable that have respectively the χ^2 distribution $\chi^2(r_1) \ldots \chi^2(r_n)$. Then the random variable $Y = X_1 + X_2 + \cdots + X_n$ has a χ^2 distribution with $r_1 + r_2 + \cdots + r_n$ degree of freedom ie) $Y \sim \chi^2(r_1 + r_2 + \cdots + r_n)$

Soln:

$$\text{Since } M_1(t) = E(e^{tx_1})$$

$$= (1-2t)^{-r/2}, \quad t < 1/2$$

Given that

x_1 is $\chi^2(r_1)$ $i = 1, 2, \ldots$

x_2 is $\chi^2(r_2)$

$\vdots$

x_n is $\chi^2(r_n)$

$$\text{mgf of } x_1 = (1-2t)^{-r_1/2}$$

$$\text{mgf of } x_2 = (1-2t)^{-r_2/2}$$

$\vdots$

$$\text{mgf of } x_n = (1-2t)^{-r_n/2}$$

$$\text{mgf of } Y = \sum_i x_i = E(e^{tY})$$

$$= E(e^{tx_1}) E(e^{tx_2}) \cdots E(e^{tx_n})$$

[$x_1, x_2, \ldots, x_n$ are independent]

$$= (1-2t)^{-r_1/2} (1-2t)^{-r_2/2} \cdots (1-2t)^{-r_n/2}$$

$$= (1-2t)^{-(r_1/2 + r_2/2 + \cdots + r_n/2)}$$

$$= (1-2t)^{-(r_1 + r_2 + \cdots + r_n)/2}$$

Y is $\chi^2 (r_1 + r_2 + \cdots + r_n)$

problem:

Let X_1, X_2, X_3 denote the random sample of size 3 from the standard normal distribution.

Let Y denote the statistic

ie) The sum of the square of the sample observation

$$Y = X_1^2 + X_2^2 + X_3^2$$

Find the p.d.f (or) prove that Y is the χ^2 distribution with 3 degree of freedom.

soln:

$$Y = X_1^2 + X_2^2 + X_3^2$$

The distribution function of Y is

$$G(y) = P(Y \leq y) \qquad \because P(X \leq x) = \int\limits_{x} f(x)\,dx$$

$$= P(X_1^2 + X_2^2 + X_3^2 \leq y)$$

If $y \leq 0 \Rightarrow G(y) = 0$

If $y \geq 0 \Rightarrow G(y) = \iiint\limits_{A} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3$

where A is the set of points x_1, x_2, x_3 which is interior to 0 or n.

The surface of sphere with centre $(0,0,0)$ and radius $\sqrt{y}$.

$$f(x_1\, x_2\, x_3) = f(x_1) \cdot f(x_2) \cdot f(x_3)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2} \frac{1}{\sqrt{2\pi}} e^{-x_3^2/2}$$

$$= \frac{1}{(2\pi)^{3/2}} e^{-(x_1^2+x_2^2+x_3^2)/2}$$

$$G(y) = \iiint_A \frac{1}{(2\pi)^{3/2}} e^{-(x_1^2+x_2^2+x_3^2)/2} dx_1 dx_2 dx_3$$

To evaluate the integral converting condition co-ordinate to spherical polar co-ordinates by taking

$$x_1 = \rho \cos\theta \sin\phi$$

$$x_2 = \rho \sin\theta \sin\phi$$

$$x_3 = \rho \cos\phi$$

where $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$ and $\rho \ge 0$

ρ varies from 0 to $\sqrt{y}$

$$dx_1\, dx_2\, dx_3 = |J|\, d\rho\, d\theta\, d\phi$$

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(\rho, \theta, \phi)}$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial \rho} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_1}{\partial \phi} \\ \frac{\partial x_2}{\partial \rho} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_2}{\partial \phi} \\ \frac{\partial x_3}{\partial \rho} & \frac{\partial x_3}{\partial \theta} & \frac{\partial x_3}{\partial \phi} \end{vmatrix}$$

$x_1 = \rho\cos\theta\sin\phi$

$\frac{\partial x_1}{\partial \rho} = \cos\theta\sin\phi$

$\frac{\partial x_1}{\partial \theta} = -\rho\sin\theta\sin\phi$

$\frac{\partial x_1}{\partial \phi} = \rho\cos\theta\cos\phi$

$$J = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\theta\sin\phi\left[-\rho^2\cos\theta\sin^2\phi\right] + \rho\sin\theta\sin\phi\left[-\rho\sin\theta\sin^2\phi - \rho\sin\theta\cos^2\phi\right] + \rho\cos\theta\cos\phi\left[-\rho\cos\theta\sin\phi\cos\phi\right]$$

$$= -\rho^2\cos^2\theta\sin^3\phi - \rho^2\sin^2\theta\sin^3\phi - \rho^2\sin^2\theta\cos^2\phi\sin\phi - \rho^2\cos^2\theta\cos^2\phi\sin\phi$$

$$= -\rho^2\sin^3\phi\left(\cos^2\theta + \sin^2\theta\right) - \rho^2\cos^2\phi\sin\phi\left(\cos^2\theta + \sin^2\theta\right)$$

$$= -\rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right)$$

$$J = -\rho^2\sin\phi$$

$$|J| = \rho^2\sin\phi$$

Since $G(y) = \iiint \frac{1}{(2\pi)^{3/2}} e^{-(x_1^2+x_2^2+x_3^2)/2}\, dx_1\, dx_2\, dx_3$

~~To evaluate the integral converting condition co-ordinate to spherical polar co-ordinate to spherical polar co-ordinate~~

Now $G(y) = \frac{1}{(2\pi)^{3/2}} \int_0^{\sqrt{y}} \int_0^{2\pi} \int_0^{\pi} e^{-\rho^2/2} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^{\sqrt{y}} \rho^2 e^{-\rho^2/2}\, d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi$$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^{\sqrt{y}} \rho^2 e^{-\rho^2/2}\, d\rho\, \left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_0^{\pi}$$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^{\sqrt{y}} \rho^2 e^{-\rho^2/2} d\rho \, [2\pi] \, [-\cos\pi + \cos 0]$$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^{\sqrt{y}} \rho^2 e^{-\rho^2/2} d\rho \, (2\pi)(1+1)$$

$$= \frac{4\pi}{(2\pi)^{3/2}} \int_0^{\sqrt{y}} \rho^2 e^{-\rho^2/2} d\rho$$

$$= \frac{2\pi}{2\pi (2\pi)^{1/2}} \int_0^{\sqrt{y}} \rho^2 e^{-\rho^2/2} d\rho$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \int_0^{\sqrt{y}} \rho^2 e^{-\rho^2/2} d\rho$$

$$= \sqrt{2/\pi} \int_0^{\sqrt{y}} \rho^2 e^{-\rho^2/2} d\rho$$

To evaluate this integral

$$\rho = \sqrt{w} = w^{1/2}$$

$$d\rho = \frac{1}{2} w^{-1/2} dw$$

$$d\rho = \frac{1}{2\sqrt{w}} dw$$

when $\rho = 0 \Rightarrow w = 0$

$\rho = \sqrt{y} \Rightarrow \sqrt{w} = \sqrt{y}$

$$G(y) = \sqrt{2/\pi} \int_0^{\sqrt{y}} w e^{-w/2} \frac{1}{2\sqrt{w}} dw$$

$$= \frac{\sqrt{2}}{2\sqrt{\pi}} \int_0^{\sqrt{y}} \sqrt{w} e^{-w/2} dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{y}} \sqrt{w} e^{-w/2} dw$$

Y is random variable of continuous type.

The p.d.f of $Y = g(y)\ G(y)$

$$G'(y) = \frac{d}{dy}[G(y)] = \frac{d}{dw} G(y) \cdot \frac{\partial w}{\partial y}$$

$$= \frac{d}{dw}\left[\frac{1}{\sqrt{2\pi}} \int_0^y \sqrt{w}\, e^{-w/2}\, dw\right]$$

$$\left[\because y = w.\ \frac{dw}{dy} = 1\right]$$

$$g(y) = \frac{1}{\sqrt{2\pi}} \sqrt{y}\, e^{-y/2} \qquad 0<y<\infty$$

We know that

chi-square distribution with r degree of freedom is

$$f(x) = \frac{1}{\Gamma r/2\ 2^{r/2}} x^{r/2}\ .\ 1.\ e^{-x/2}$$

$$\Gamma \alpha = (\alpha-1)\Gamma \alpha-1$$

$$\Gamma 3/2 = (3/2 - 1)\Gamma 3/2-1$$

$$= \frac{1}{2}\Gamma 1/2$$

$$= \frac{1}{2}\sqrt{\pi}$$

$$\sqrt{\pi} = 2\Gamma 3/2$$

$$g(y) = \frac{1}{\Gamma 3/2\ 2^{3/2}}\ y^{3/2-1}\ e^{-y/2}$$

$$g(y) = \begin{cases} \frac{1}{\Gamma 2\ 2\Gamma 3/2}\ y^{3/2-1}\ e^{-y/2}, & 0<y<\infty \\ 0 & \text{otherwise} \end{cases}$$

$\therefore$ y has chi-square distribution with 3 degrees of freedom

UNIT - V

LIMITTIG DISTRIBUTIONS

Unit - V

Limitting distribution

Let the distribution function $f_n(y)$ of the random variable Y_n depend upon n $n=1,2,3,\ldots$ If $F(y)$ is a distribution function and if $\lim_{n\to\infty} F_n(y) = f(y)$ for every point y at which $F(y)$ is continuous then the sequence of random variable $Y_1, Y_2, \ldots$ converges is distribution to a random variable with distribution function $F(y)$.

The following example are illustrative of this converges in distribution.

Example: 1

Let Y_n denote the n^{th} order statistic of a random sample $X_1, X_2, \ldots, X_n$ from a distribution having p.d.f

$$f(x) = \begin{cases} 1/\theta & 0 < x < \theta,\ 0 < \theta < \infty \\ 0 & \text{elsewhere} \end{cases}$$

The pdf of Y_n is

$$g(y) = \begin{cases} ny^{n-1} & 0 < y < \theta \\ 0 & \text{elsewhere} \end{cases}$$

and the distribution function of Y_n is

$F_n(y) = 0,\ y < \theta$

$$F_n(y) = \int_0^y \frac{ny^{n-1}}{\theta^n}\,dy = \frac{n}{\theta^n}\left[\frac{y^{n-1+1}}{n-1+1}\right]_0^y$$

$$= \left(\frac{y}{\theta}\right)^n, \quad 0 \le y \le \theta$$

$$F_n(y) = 1, \quad 0 \le y < \infty$$

Then $\lim\limits_{n\to\infty} F_n(y) = 0 \quad -\infty < y < \theta$

$\lim\limits_{n\to\infty} F_n'(y) = 1 \quad \theta \le y < \infty$

Now $F(y) = 0 \;, \quad -\infty < y < \theta$

$F(y) = 1 \;, \quad \theta \le y < \infty$ is a distril function

more over

$\lim\limits_{n\to\infty} F_n(y) = F(y)$ at each point of continuity of $F(y)$

A distribution of the discrete type which has a probability of 1 atleast a single point has been called a degenerate distribution.

Thus in this example, the sequence of the n^{th} order statistics $Y_n, n=1,2,\ldots$ converges in distribution to a random variable that has a degenerate distribution at the point $y=\theta$.

Example: 2

Let $\bar{X}_n$ have the distribution function

$$f_n(\bar{x}) = \int_{-\infty}^{\bar{x}} \frac{1}{\sqrt{\frac{1}{n}}\sqrt{2\pi}} e^{-nw^2/2} \, dw$$

If the change of variable $v = \sqrt{u}\, w$ is made we have

$$F_n(\bar{x}) = \int_{-\infty}^{\sqrt{n}\bar{x}} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} \, dv$$

It is clear that

$$\lim_{n\to\infty} F_n(\bar{x}) = 0 ; \quad \bar{x} < 0$$

$$\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$$

put $v^2/2 = t \Rightarrow v = \sqrt{2t}$

$$2v\,dv = 2\,dt \Rightarrow v\,dv = dt$$

$$\sqrt{2t}\,dv = dt$$

$$dv = \frac{dt}{\sqrt{2t}}$$

$v = -\infty \qquad t = \infty$

$v = \sqrt{n}\,\bar{x} \qquad t = \frac{n\bar{x}^2}{2}$

$$= \int_{\frac{n\bar{x}^2}{2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \int_{\frac{n\bar{x}^2}{2}}^{\infty} \frac{1}{2\sqrt{\pi}} e^{-t} t^{-1/2+1-1} dt$$

$$= \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} t^{1/2-1} e^{-t} dt$$

$$= \frac{1}{2\sqrt{\pi}} \Gamma(1/2)$$

$$= \frac{1}{2\sqrt{\pi}} \sqrt{\pi}$$

$$= 1/2$$

$\lim_{n\to\infty} F_n(\bar{x}) = \frac{1}{2} \quad \bar{x} = 0 = 1 \quad , \bar{x} > 0$

Now the function

$F(\bar{x}) = 0 \quad , \quad \bar{x} > 0$

$= 1 \quad \bar{x} > 0$ is a distribution function and $\lim_{n\to\infty} F_n(\bar{x}) = F(\bar{x})$ at every point of continuity of $F(\bar{x})$

To be sure $\lim_{n\to\infty} F_n(0) \neq F(0)$

but $F(\bar{x})$ is not continuous at $\bar{x}=0$

According the sequence $\bar{x}_1, \bar{x}_2, \ldots$ converges in distribution to a random variable that has a degenerate distribution at $\bar{x}=0$.

Example: 3

Even if a sequence $X_1, X_2, \ldots, X_n$ converges in distribution to a r.v X we cannot in general determine the distribution of X by taking the limit of the pdf of X_n.

This is illustrated by taking X_n have the pdf

$$f_n(x) = 1 \quad x = 2 + 1/n$$
$$= 0 \quad \text{otherwise}$$

clearly $\lim_{n\to\infty} F_n(x) = 0 \;\forall$ values of x.

This may suggest that $x_n, n=1,2,3,\ldots$ does not converges is distribution

However the distribution function of X_n is

$$f_n(x) = 0, \quad x < 2 + 1/n$$
$$= 1, \quad x \geq 2 + 1/n \text{ and}$$

$$\lim_{n\to\infty} f_n(x) = 0, \quad x \leq 2$$
$$= 1, \quad x > 2$$

since $F(x) = 0, \quad x < 2$

$$= 1, \quad x \geq 2$$

is a distribution function and since $\lim_{n\to\infty} F_n(x) = F(x)$ at all points of continuity of $f(x)$, the sequence $x_1, x_2, \dots$ converges in distribution to a r.v with distribution $f(x)$

$$\lim_{n\to\infty} F_n(x) = F(x) \text{ at all points of } x$$

for which $F(x)$ is continuous for that reason we after find it converient to refer to $f(x)$ as the limitting distribution

Example : 4

Let Y_n denote the n^{th} order statistic of a random sample from the uniform distribution of Ex : 1

Let $Z_n = n(\theta - Y_n)$. The pdf of Z_n

$$h_n(z) = \frac{(\theta - z/n)^{n-1}}{\theta^n}, \quad 0 < z < n\theta$$

0 elsewhere and the distribution function of Z_n is $G_n(z) = 0, z < 0$

$$= \int_0^z \frac{(\theta - w/n)^{n-1}}{\theta^n} dw$$

$$= 1 - \left(1 - \frac{z}{n\theta}\right)^n, \quad 0 \le z \le n\theta$$

$$= 1, \quad n\theta \le z$$

Hence $\lim_{n\to\infty} G_n(z) = 0, \ z \le 0$

$$= 1 - e^{-z/\theta}, \quad 0 < z < \infty$$

Now $G(z) = 0, \ z < 0$

$= 1 - e^{-z/\theta}, \ 0 < z$ is a distribution function that is everywhere continuous and $\lim_{n\to\infty} G_n(z) = G(z)$ at all points.

Thus Z_n has a limitting distribution with distribution function $G(z)$. This affords us an example of a limitting distribution that is not degenerate.

Example : 5

Let T_n have a t-distribution with n degree of freedom $n = 1, 2, \ldots$ Thus its distribution function is

$$F_n(t) = \int_{-\infty}^{t} \frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\,\Gamma n/2} \frac{1}{(1+y^2/n)^{(n+1/2)}} dy$$

where the integrant is the p.d.f $f_n(y)$ of T_n. Accordingly

$$\lim_{n\to\infty} F_n(t) = \lim_{n\to\infty} \int_{-\infty}^{t} f_n(y)\, dy = \int_{-\infty}^{t} \lim_{n\to\infty} F_n(y)\, dy$$

The change of the order of the limit and integration is justified because $|f_n(y)|$ is determined by a function like $10 f_1(y)$ with a finite integral that is $|f_n(y)| \leq 10 f_1(y)$ and

$$\int_{-\infty}^{t} 10 f_1(y)\, dy = \frac{10}{\pi} \arc\tan t < \infty \ \forall$$

real t.

Hence here we can find the limitting distribution by finding the limit of the pdf of T_n

It is $\lim_{n\to\infty} f_n(y) = \lim_{n\to\infty}\left[\frac{\Gamma\overline{n+1/2}}{\sqrt{n/2}\,\Gamma\overline{n/2}}\right] \times$

$$\lim_{n\to\infty}\left[\frac{1}{(1+y^2/n)^{1/2}}\right] \lim_{n\to\infty}\left\{\frac{1}{\sqrt{2\pi}}\left[1+y^2/n\right]^{-n/2}\right\}.$$

Using the fact from elementary calculus that $\lim_{n\to\infty}(1+y^2/n)^n = e^{-y^2}$

The limit associated with the 3rd factor is clearly the p.d.f of the standard normal distribution.

The second limit obviously equals 1) If we know more about the gamma it is easy to show that the first limit also equal 1

Thus we have

$\lim_{n\to\infty} F_n(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-y^2} dy$ and hence T_n has a limitting standard normal distribution.

Defn:

A sequence r.v $x_1, x_2, \ldots$ converges is probability to a r.v x. If for every $\varepsilon > 0$

$\lim_{n\to\infty} Pr(|x_n - x| \not< \varepsilon) = 1$ or equivalently

$\lim_{n\to\infty} Pr(|x_n - x| < \varepsilon) = 0$ statisticians are usually interested in this convergence when the r.v x has a degenerate distribution at that constant. Hence we concentrate on that situation.

Eg: 1

Of size n from a distribution that has mean μ and positive variance σ^2. Then the mean and variance of $\bar{x}_n$ are μ and σ^2/n

consider for every fixed $\varepsilon > 0$ then probability $Pr(|\bar{x}_n - \mu| \geq \varepsilon) = Pr\left(|\bar{x}_n - \varepsilon| \geq \frac{k\sigma}{\sqrt{n}}\right)$ where $k = \varepsilon\sqrt{n}/\sigma$. In accordance with is less than or equal to $\frac{1}{k^2} = \sigma^2/n\varepsilon^2$. So for every fixed $\varepsilon > 0$, we have

$$\lim_{n\to\infty} Pr(|\bar{x}_n - \mu| \geq \varepsilon) \leq \lim_{n\to\infty} \sigma^2/nc^2 = 0$$

Hence $\bar{x}_n$, $n = 1, 2, \ldots$ converges in probability to μ if σ^2 is finite. This result is called weak law of large number.

Remark:

A stronger type of converges is given by $Pr(\lim_{n\to\infty} Y_n = c) = 1$ in this case we say that Y_n, $n = 1, 2, \ldots$ convergence to c with probability 1.

Although we do not consider this type of convergence it is known that mean $\bar{x}_n$ $n = 1, 2, \ldots$ of a random sample converges with probability 1 to the mean μ of the distribution provided that the latter exist this is one form of the strong law of large number.

Thm: 1

Let $F_n(y)$ denote the distribution of r.v Y_n whose distribution depends upon the positive integers n. Let c denote a constant which does not depend upon n. The sequence Y_n, $n=1,2,\dots$ converges probability to the constant c $\Leftrightarrow$ the limitting distribution of Y_n is degenerate at $Y=c$.

proof:- First assume that the

$$\lim_{n\to\infty} Pr(|Y_n - c| < \varepsilon) = 1 \text{ for every } \varepsilon > 0$$

We are to prove that the r.v Y_n such that

$$\lim_{n\to\infty} F_n(Y) = 0, \quad Y < c$$
$$= 1, \quad Y > c$$

Note that we do not need to know anything about $\lim_{n\to\infty} F_n(c)$

For if the lim of $F_n(y)$ is as indicated.

Then Y_n has a limitting distribution with distribution function

$$F_n(y) = 0, \quad y < c$$
$$= 1, \quad y \geq c$$

Now $Pr(|Y_n - c| < \varepsilon) = F_n[(c+\varepsilon)-] - F_n(c-\varepsilon)$

where $F_n[(c+\varepsilon)-]$ is the left-hand limit of $F_n(y)$ at $y = c+\varepsilon$

Thus we have $1 = \lim_{n\to\infty} Pr(|Y_n - c| < \varepsilon)$

$$= \lim_{n\to\infty} F_n[(c+\varepsilon)-] - \lim_{n\to\infty} F_n(c-\varepsilon)$$

Because $0 \leq F_n(y) \leq 1$ $\forall$ values of Y and for every positive integers n, it must be that

$\lim_{n\to\infty} F_n(c-\varepsilon)=0$, $\lim_{n\to\infty} F_n[(c+\varepsilon)-]=1$

Since this is true for every $\varepsilon>0$ we have $\lim_{n\to\infty} F_n(y)=0$, $y<c$

$=1$, $y>c$ as we are to prove that

$$\lim_{n\to\infty} Pr(|Y_n-c|<\varepsilon)=1 \quad \forall\ \varepsilon>0$$

Because $Pr(|Y_n-c|<\varepsilon)=F_n[(c+\varepsilon)-]-F_n(c-\varepsilon)$

and because it is given that

$$\lim_{n\to\infty} F_n[(c+\varepsilon)-]=1$$

$$\lim_{n\to\infty} F_n(c-\varepsilon)=0 \quad \forall\ \varepsilon>0$$

We have the desired result

The complete the proof.

Thm: 2

Let the r.v Y_n have the distribution function $F_n(y)$ and the mgf $M(t:n)$ there exist for $-n<t-n$ $\forall$ n. If there exist a distribution function $F(y)$ with corresponding moment generating function $M(t)$ defined for $|t|\le n, <n$ such that $\lim_{n\to\infty} M(t:n)=M(t)$ then Y_n has a limitting distribution with distribution function $F(y)$.

proof:-

We refer to a limit of the form

$\lim_{n\to\infty}\left[1+\frac{b}{n}+\frac{\chi(n)}{n}\right]^{cn}$ where b and c do not depend upon n and where $\lim_{n\to\infty}\chi(n)=0$

Then

$$\lim_{n\to\infty}\left[1+b/n+\frac{\chi(n)}{n}\right]^{cn} = \lim_{n\to\infty}[1+b/n]^{cn}$$

$$= e^{bc}$$

For example

$$\lim_{n\to\infty}\left[1-\frac{t^2}{n}+\frac{t^3}{n^{3/2}}\right]^{-n/2} = \lim_{n\to\infty}\left[1-t^2/n+\frac{t^3/\sqrt{n}}{n}\right]^{-n/2}$$

Here $b=-t^2$, $c=-1/2$ and $\chi(n)=t^3/\sqrt{n}$

Accordingly, for every fixed value of t, the limit is $e^{t^2/2}$.

Example

U.Q 5m

Let Z_n be $\chi^2(n)$. Then the m.g.f of Z_n is $(1-2t)^{-n/2}$, $t<1/2$. The mean and variance of Z_n are respectively n and $2n$. The limitting distribution of the r.v $Y_n = (Z_n-n)/\sqrt{2n}$ will be investicated.

Now the mgf of Y_n is

$$M(t:n) = E\left\{\exp\left[t\left(\frac{Z_n-n}{\sqrt{2n}}\right)\right]\right\}$$

$$= e^{-tn/\sqrt{2n}}\, E\left(e^{tZ_n/\sqrt{2n}}\right)$$

$$= \exp\left[-\left(t\sqrt{2}/n\right)\left(n/2\right)\right]\left[1-\frac{2t}{\sqrt{2n}}\right]^{-n/2},$$

$$t<\frac{\sqrt{2n}}{2}$$

This may be written in the form

$$M(t:n) = \left(e^{t\frac{\sqrt{2}}{n}} - t\sqrt{\tfrac{2}{n}}\, e^{t\sqrt{2/n}}\right)^{-n/2}, \quad t<\sqrt{n/2}$$

In accordance with Taylor's formula there exist a number $\varepsilon(n)$ between 0 and $t\sqrt{2/n}$ such that

$$e^{t\sqrt{2/n}} = 1 + t\sqrt{2/n} + \frac{1}{2}\left(t\sqrt{2/n}\right)^2 + \frac{e^{\varepsilon(n)}}{6}\left(t\sqrt{2/n}\right)^3$$

If this sum is substituted for $e^{t\sqrt{2/n}}$ in the last expression for $M(t:n)$ it is seen that

$$M(t:n) = \left(1 - \frac{t^2}{n} + \frac{\chi(n)}{n}\right)^{-n/2}$$

where $\chi(n) = \frac{\sqrt{2}\,t^3 e^{\varepsilon(n)}}{3\sqrt{n}} - \frac{\sqrt{2}\,t^3}{\sqrt{n}} - \frac{2t^4 e^{\varepsilon(n)}}{3n}$

Since $\varepsilon(n) \to 0$ as $n \to \infty$. Then $\lim \chi(n) = 0$ for every fixed value of t.

In accordance with the limit proposition cited earlier in this section we have

$\lim\limits_{n\to\infty} M(t:n) = e^{t^2/2}$

For all real value of t, that is r.v $Y_n = (X_n - n)\sqrt{2n}$ has a limitting standard normal distribution.

The central limit thm:

10m

If $X_1, X_2, \dots, X_n$ is a random sample from a normal distribution with mean μ and variance σ^2 of the r.v

$$\sum_{i}^{n} \frac{x_i - n\mu}{\sigma\sqrt{n}} = \frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma}$$ is for every positive integers n normally distributed with zero mean and unit variance. In probability theory there is a very elegant

theorem called the central limit theorem.

Thm: Let $x_1, x_2, \ldots, x_n$ denote the observations of a random sample from a distribution that has mean μ and positive variance σ^2. Then the r.v $Y_n = (\sum_i^n x_i - n\mu)/\sqrt{n}\sigma = \sqrt{n}(\bar{x}_n - \mu)/\sigma$ has a limitting distribution that is normal with mean zero and variance.

proof:

We assume the existance of the mgf $M(t) = E(e^{tx})$, $-h<t<h$ of the distribution.

However this proof of essentially the same one that would be given for this theorem in more advanced course by replacing the m.g.f by the characteristic function $\phi(t) = E(e^{itx})$

The function $M(t) = E[e^{t(x-\mu)}]$

$= e^{-\mu t} M(t)$ also exist

for $-h<t<h$

Since $M(t)$ is the m.g.f for $x-\mu$ is must follow that $M(0)=1$, $M'(0) = E(x-\mu) = 0$ and $M''(0) = E(x-\mu)^2 = \sigma^2$

By Taylors formula there exist a number between 0 and 1 such that

$$M(t) = M(0) + M'(0) + \frac{M''(\varepsilon)t^2}{2}$$

$$= 1 + \frac{M''(\varepsilon)t^2}{2}$$

If $\sigma^2 t^2/2$ is added and subtracted then

$$M(t) = 1 + \frac{\sigma^2 t^2}{2} + \frac{[M''(\varepsilon) - \sigma^2]t^2}{2} \quad (1)$$

Next consider $M(t:n)$

where $M(t:n) = E\left[\exp\left(t \frac{\Sigma x_i - n\mu}{\sigma\sqrt{n}}\right)\right]$

$$E\left[\exp\left(t \frac{x_1-\mu}{\sigma\sqrt{n}}\right) - \exp\left(t \frac{x_2-\mu}{\sigma\sqrt{n}}\right) \dots \exp\left(t \frac{x_n-\mu}{\sigma\sqrt{n}}\right)\right]$$

$$= E\left[\exp\left(t \frac{x_1-\mu}{\sigma\sqrt{n}}\right)\right] \dots E\left[\exp\left(t \frac{x_n-\mu}{\sigma\sqrt{n}}\right)\right]$$

$$= \left\{E\left(\exp\left(t \frac{x-\mu}{\sigma\sqrt{n}}\right)\right)\right\}^n$$

$$= \left[M\left(t/\sigma\sqrt{n}\right)\right]^n, \quad -n < t/\sigma\sqrt{n} < n$$

In eqn (1) replace t by $t/\sigma\sqrt{n}$ to obtain

$$M\left(t/\sigma\sqrt{n}\right) = 1 + \frac{t^2}{2n} + \frac{[M''(\varepsilon) - \sigma^2]t^2}{2n\sigma^2}$$

where now ε is between 0 and $\frac{t^2}{\sigma\sqrt{n}}$

with $-n\sigma\sqrt{n} < t < n\sigma\sqrt{n}$

Accordingly

$$M(t:n) = \left\{1 + \frac{t^2}{2n} + \frac{[M''(\varepsilon) - \sigma^2]t^2}{2n\sigma^2}\right\}^n$$

Since $M''(t)$ is continuous at $t=0$ and since $\varepsilon \to 0$ as $n \to \infty$ we have

$$\lim_{n\to\infty} [M''(\varepsilon) - \sigma^2] = 0$$

The limit proposition cited to show that $\lim_{n\to\infty} M(t:n) = e^{t^2/2}$ $\forall$ real ~~numbers~~ values of t.

This proves that the random variable $Y_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$ has a limitting standard normal distribution.